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The Maximum Number of Maximal Independent Sets in Forests[1]

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Abstract

Let G be a simple and undirected graph. By $mi(G)$ we denote the number of maximal independent sets in G. In this note, we determine the maximum number of maximal independent sets among the set of a type of forests.

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1 Introduction

Let G be a graph with vertex set and edge set being $V(G)$ and $E(G)$, respectively. A subset S of $V(G)$ is said to be an independent set of G if any two vertices in S are not adjacent in G. We call an independent set *maximal* if it is not a proper subset of any other independent set of G. Let $mi(G)$ denote the number of all maximal independent sets in G. For all notations and terminology not defined here, we follow that of [1].

Around 1960, Erdö and Moser proposed the problem of determining the maximum number of $mi(G)$ in the family of graphs of order n and characterizing structure of graphs attaining the maximum value. Shortly after, Moon and Moser [2] solved the problem. The same problem was further investigated for certain families of graphs. Furedi [3] and, independently, Griggs et al. [4] studied the problem for the family of connected graphs. By employing different techniques, Wilf [5], Sagan [7] and Jou and Chang [6] solved the problem for trees. Jou and Chang [6] further explored the problem for forests and also for graphs with at most one cycle. Hujter and Tuza [8] determined the maximum

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value of $mi(G)$ for the family of triangle-free graphs. As for the connected triangle-free graphs, it was solved by Jou and Chang [9]. Recently, Sagan and Vatter [10] settled the problem for the family of graphs with at most r cycles and its connected counterpart, and Goh et al. [11] independently obtained the same result under the condition that the order of the graph is no less than $3r$. Recently, Jin and Li [12] investigated the second largest cardinality of $mi(G)$ among all graphs of order n. More recently, Koh et al. [13] determined the maximum number of $mi(G)$ over the set of unicyclic connected graphs. In the same paper, they also proved an interesting result, that is, if G is a forest of order n and contains a path P_r ($n \geq r \geq 2$), then $mi(G) \leq 2^k mi(P_r)$ if $n - r = 2k$, and $mi(G) \leq 2^k mi(P_{r+1})$ if $n - r = 2k + 1$.

In this note, we determine the maximum number of maximal independent sets among the set of a type of forests.

2 Lemmas and result

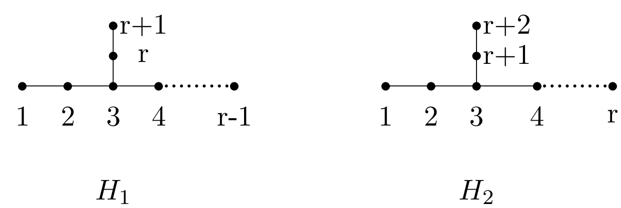

Fig. 1.

We begin with some preliminary results, which will be helpful to the proof of our main result.

Lemma 2.1 (Hujter and Tuza [8]). *Let x be any vertex in a graph G. Then*

(i). $mi(G) \leq mi(G - x) + mi(G - N[x])$;

(ii). If x is a leaf adjacent to the vertex y, then $mi(G) = mi(G - N[x]) + mi(G - N[y])$.

Lemma 2.2 (Hujter and Tuza [8]). *For any two vertex-disjoint graphs G_1 and G_2, $mi(G_1 \cup G_2) = mi(G_1)mi(G_2)$.*

Lemma 2.3 (Furedi [3]). *For $n \geq 4$, $mi(P_n) = mi(P_{n-2}) + mi(P_{n-3})$.*

Corollary 2.4 (Koh et al. [13]). *(i).For $n \geq 3$, $mi(P_n) \leq 2mi(P_{n-2})$ with the equality if and only if $n \in \{3, 5\}$,* *(ii). For $n \geq 6$, $mi(P_n) = mi(P_{n-1}) + mi(P_{n-5})$.*

By means of Lemmas 2.1–2.3 and Corollary 2.4, one can easily prove the following:

Lemma 2.5. *(i). For $n \geq 6$, $mi(T_n^3) = mi(P_{n-2}) + mi(P_{n-4})$.*
(ii).For $n \geq 8$, $mi(T_n^3) \leq 2mi(T_{n-2}^3)$.

Lemma 2.6(Koh et al. [13]). *If G is a caterpillar of order n, then $mi(G) \leq mi(P_n)$ with the equality holding if and only if $G \cong P_n$ or T_6^3.*

Two vertices x and y are said to be *duplicated leaves* in G, if they are leaves of the same vertex, say z, in G.

Lemma 2.7(Hujter and Tuza [8]). *If x and y are duplicated leaves in G, then $mi(G) = mi(G - x) = mi(G - y)$.*

Theorem 2.8. *If G is a caterpillar of order $n \geq 6$ not isomorphic to P_n, then $mi(G) \leq mi(T_n^3)$.*

Proof. We shall prove the theorem by induction on n.

If $n = 6$, the result is obviously valid by Lemma 2.6. Suppose that the result is true for all caterpillars of order $n^{'} < n$. Assume now that G is a caterpillar of $n \geq 7$ vertices and not isomorphic to T_n^3.

Take from G a longest path $P_l = x_1 \cdots x_l$. Assume first that $l \leq 5$. If $l \leq 4$, since G is a caterpillar and $n \geq 7$, G must have duplicated leaves, then by Lemmas 2.5, 2.6 and 2.7, we have $mi(G) = mi(P_l) \leq mi(P_4) < mi(P_{n-2}) + mi(P_{n-4}) = mi(T_n^3)$.

If $l = 5$, since G is a caterpillar not isomorphic to P_n and $n \geq 7$, G must have duplicated leaves, then by Lemma 2.7, we have $mi(G) = mi(T_6^3)$ or $mi(P_5)$. In either cases, we have $mi(G) < mi(T_n^3)$.

Assume that $l \geq 6$ in what follows.

If $d(x_2) \geq 3$, G has duplicated leaves (say x_1), and thus by Lemma 2.6, $mi(G) = mi(G - x_1) \leq mi(P_{n-1}) = mi(P_{n-3}) + mi(P_{n-4}) < mi(P_{n-2}) + mi(P_{n-4}) = mi(T_n^3)$, and the result follows.

So we may suppose that $d(x_2) = 2$.

If $d(x_3) \geq 4$, G has duplicated leaves, and by the way similar to above, we can show that $mi(G) < mi(T_n^3)$.

If $d(x_3) = 3$, by Lemmas 2.1 and 2.6, it must hold that

$$\begin{aligned} mi(G) &= mi(G - x_1 - x_2) + mi(G - x_1 - x_2 - x_3) \\ &\leq mi(P_{n-2}) + mi(P_{n-4}) \\ &= mi(T_n^3). \end{aligned}$$

Now consider the case that $d(x_3) = 2$. By symmetry, we may assume that $d(x_l) = d(x_{l-1}) = d(x_{l-2}) = 2$. Note that $G \ncong P_n$, we may assume that $l \geq 7$ in the following.

If $d(x_4) \geq 4$, G has duplicated leaves, since G is a caterpillar. By a reasoning similar to above, the result holds.

If $d(x_4) = 3$, let $y \in N(x_4) \setminus \{x_3, x_5\}$, by Lemmas 2.1 and 2.6 and induction hypothesis ,

$$\begin{aligned} mi(G) &= mi(G - N[y]) + mi(G - N[x_4]) \\ &\leq mi(P_3)mi(P_{n-5}) + mi(P_2)mi(P_{n-6}) \\ &= 2mi(P_{n-5}) + 2mi(P_{n-6}) \\ &= 2mi(P_{n-3}) \\ &\leq mi(P_{n-2}) + mi(P_{n-4}) \\ &= mi(T_n^3). \end{aligned}$$

Now, we may suppose that $d(x_4) = 2$. From above discussions, we may assume that $d(x_l) = d(x_{l-1}) = d(x_{l-2}) = d(x_{l-3}) = 2$. Note that $G \ncong P_n$. So we have $l \geq 9$ and $n \geq l + 1 \geq 10$. Since $G - x_1 - x_2 \ncong P_{n-2}$ and $G - x_1 - x_2 - x_3 \ncong P_{n-3}$, for otherwise, $G \cong P_n$, then by induction assumption and Lemma 2.1, $mi(G) \leq mi(T^3_{n-2}) + mi(T^3_{n-3}) = mi(T^3_n)$. This completes the proof. □

For $n \geq r \geq 6$, let $\mathrm{M}(r;n) = \begin{cases} 2^k mi(T_r^3) & if\ n = 2k + r; \\ 2^k mi(T_{r+1}^3) & if\ n = 2k + r + 1. \end{cases}$

It is easy to prove that $\mathrm{M}(r;n)$ has the following properties, whose proofs we omit here.

Lemma 2.9. *For any integers $n \geq r \geq 6$,*
(i). $\mathrm{M}(r;n) \leq \mathrm{M}(r;n+1)$,
(ii).$2^t\mathrm{M}(r;n) = \mathrm{M}(r;n+2t)$, *and*
(iii).$m(T_n^3) \leq \mathrm{M}(r;n)$.

Lemma 2.10. (Wilf [5])*For any forest G of order n, $mi(G) \leq 2^{\lfloor \frac{n}{2} \rfloor}$. Furthermore, if $n = 2k$, then the equality holds if and only if G is the disjoint union of k copies of K_2.*

For a connected graph G, if S is the subset of $V(G)$ and x is a vertex in $V(G)$, we let $d(x, S) = \min_{y \in S} d(x, y)$. It is obvious that $d(x, S) = 0$ if and only if $x \in S$. Moreover, if $T = G[S]$, then we write $d(x, T)$ for $d(x, S)$. Furthermore, for any induced subgraph T of G, let $d^{(*)}(T) = \max_{x \in V(G)} d(x, T)$. By the above notation, we find that if G is a caterpillar, then for any longest path P in G, we have $d^{(*)}(P) \leq 1$.

Theorem 2.11. *Let n and r be integers with $n \geq r \geq 8$ and G is a forest on n vertices containing the subtree T_r^3. If $G \ncong H_1, H_2$(see $Fig.$1.), then $mi(G) \leq \mathrm{M}(r, n)$.*

Proof. When $n = r$, the result is immediate from Lemma 2.9(iii). So we may suppose that $n \geq r + 1$ in what follows. By contradiction. Assume to the contrary that the result is not true. Let n be the minimum integer such that there exists a forest G containing T_r^3, but $mi(G) > \mathrm{M}(r, n)$. We first prove the following two claims.

Claim 1. *G does not contain duplicated leaves.*

Proof. If not so, let x be one of duplicated leaves. Then by Lemma 2.7, we have $mi(G) = mi(G - x)$. Note that $G - x$ still contains T_r^3, thus by the choice of n and Lemma 2.9, we have $mi(G) = mi(G - x) \leq \mathrm{M}(r; n - 1) < \mathrm{M}(r; n)$, a contradiction. □

Claim 2. *G is connected.*

Proof. If not so, let $G = \bigcup_{i=1}^{s} G_i (s \geq 2)$. Suppose without loss of generality that G_1 contains T_r^3. Let n_1 denote the number of vertices in G_1. By the choice of n, we have $mi(G_1) \leq \mathrm{M}(r; n_1)$. Combining this fact and Lemmas 2.2 and 2.10, $mi(G) = mi(G_1) mi(\bigcup_{i=2}^{s} G_i) \leq \mathrm{M}(r; n_1) 2^{\lfloor \frac{n-n_1}{2} \rfloor} \leq \mathrm{M}(r; n)$, a contradiction. □

By Claims 1 and 2, we may assume now that G is connected and contains no duplicated leaves.

First assume that G is a caterpillar. Since G contains T_r^3, then $G \ncong P_n$, and thus by Theorem 2.8 and Lemma 2.9, we have $mi(G) \leq mi(T_n^3) \leq \mathrm{M}(r; n)$, a contradiction.

Now, we assume that G is a tree not isomorphic to a caterpillar. Thus $d^*(P_{r-1}) \geq 2$, where P_{r-1} is a subtree of G. If there exists some leave y in G such that $d(y, P_{r-1}) = d^*(P_{r-1})$ and $G - N[y]$ contains T_r^3, then by the choice of n, we have $mi(G - N[y]) \leq \mathrm{M}(r; n - 2)$. Let $x \in N(y)$, then

$$mi(G) = mi(G - N[y]) + mi(G - N[x]) \leq 2mi(G - N[y]) \leq 2\mathrm{M}(r; n - 2) = \mathrm{M}(r; n),$$

a contradiction.

Now, suppose that for any leave z in the set $\{z | d(z, P_{r-1}) = d^*(P_{r-1})\}$, $G - N[z]$ does not contain T_r^3. Then there must exist exactly one leave z such that $d(z, P_{r-1}) = d^*(P_{r-1})$. As G is not isomorphic to $H_i (i = 1, 2)$, G must be the graph in which the removal of z and its only neighbor results in a caterpillar of order $n - 2$, which is not isomorphic to P_{n-2}.

Let w be the unique neighbor of z. Then by Lemmas 2.1, 2.9 and Theorem 2.8,

$$\begin{aligned} mi(G) &= mi(G-N[z])+mi(G-N[w]) \\ &\leq 2mi(G-N[z]) \\ &\leq 2mi(T^3_{n-2}) \\ &\leq 2\mathrm{M}(r;n-2) \\ &= \mathrm{M}(r;n), \end{aligned}$$

a contradiction once again.

Combining all possible cases, we have completed the proof here. □

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