

## Fuzzy Queues with Priority Discipline

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### Abstract

Priority queuing models have a wide range of application in practical situations and are frequently found in computer network system. In this paper, the priority disciplined queuing models are described by using fuzzy set theory. It optimizes a fuzzy priority queuing models (preemptive priority, non preemptive priority) in which the arrival rate, service rate are fuzzy numbers. Approximate method of Extension namely DSW (Dong, Shah and Wong) algorithm is used to define membership functions of the performance measures of priority queuing system. DSW algorithm is based on the  $\alpha$  cut representation of fuzzy sets in a standard interval analysis. The performance measures obtained for the Priority queuing models are fuzzy subsets containing the whole initial information. Since the priority queues with uncertain data have more use and have broader range of applications. Numerical example is illustrated to check the validity of the proposed method.

**Keywords:** Fuzzy set theory, Queuing models, Priority discipline, DSW algorithm

### 1. Introduction

Queuing models considered have had the property that unit proceed to service on a first come - first served basis. This is obviously not only the manner of service and there are many alternatives such as last come - first served, selection in random order and selection by priority.

In priority schemes, customers with the highest priority are selected for service ahead of those with lower priority, independent of their time of arrival into the system. There are two further refinements possible in priority situation, namely preemption and non-preemption. In preemptive cases the customer with the highest priority is allowed to enter service immediately even if another with lower priority is already present in service when the higher customer arrives to the

system. In addition, a decision has to be made whether to continue the preempted customers service from the point of preemption when resumed or to start a new. The priority discipline is said to be non-preemptive if there is no interruption and the highest priority customer just goes to the head of the queue to wait his turn.

In practical, the priority queuing model, the input data arrival rate, service rate are uncertainly known. Uncertainty is resolved by using fuzzy set theory. Hence the classical queuing model with priority discipline will have more application if it is expanded using fuzzy models.

Fuzzy queuing models have been described by such researchers like Li and Lee [11], Buckley [2], Negi and Lee [12], Kao et al [10], Chen [4,5] are analyzed fuzzy queues using Zadeh's extension principle [15]. Kao et al constructed the membership functions of the system characteristic for fuzzy queues using parametric linear programming. Recently, Chen [4] developed (FM/FM/1) : ( $\infty$  / FCFS ) and (FM/FM<sup>(k)</sup>/1) : ( $\infty$  / FCFS )

## 2. Priority queues

Consider a priority queuing system with single server , infinite calling population ,in which the rate of arrival  $\tilde{\lambda}$  and rate of service  $\tilde{\mu}$ . To establish the priority discipline fuzzy queuing model, we must compare the average total cost of inactivity for the three cases : no priority discipline , Preemption priority, non preemptive priority discipline, which are denoted respectively by  $C$  ,  $C^1$  and  $C^2$ .

(a) No priority queuing model :

Average total cost of inactivity when there is no priority discipline,  $C$

$$C = (C_1 \lambda_1 + C_2 \lambda_2) W, \text{ with } W = \frac{1}{\mu - \lambda}$$

(b) Preemption priority queuing model :

Average total cost of inactivity when there is Preemption priority  $C^1$

$$C^1 = C_1 \lambda_1 W_1 + C_2 \lambda_2 W_2, \text{ with } W_{q,i} = \frac{\lambda}{\mu^2 (1-\sigma_i)(1-\sigma_{i+1})}$$

$$\sigma_1 = \frac{\lambda_1}{\mu}, \sigma_2 = \frac{\lambda_2}{\mu} \text{ and } \sigma_3 = 0$$

(c) Average total cost of inactivity when there is a non preemptive priority discipline :

Average total cost of inactivity when there is non Preemption priority  $\tilde{C}^2$

$$C^2 = C_1 \lambda_1 W_1 + C_2 \lambda_2 W_2$$

$$\text{Where } W_i = W_{q,i} + \frac{1}{\mu}, \quad W_{q,i} = \frac{1}{\mu} \frac{1}{(1-\sigma_i)(1-\sigma_{i+1})} - \frac{1}{\mu}$$

$$\sigma_1 = \frac{\lambda_1}{\mu}, \sigma_2 = \frac{\lambda_2}{\mu} \text{ and } \sigma_3 = 0,$$

$$W_1 = \frac{1}{\mu (1 - (\lambda_1/\mu)(1 - (\lambda_2/\mu))}, W_2 = \frac{1}{\mu (1 - (\lambda_2/\mu))}; L_1 = \lambda_1 W_1; L_2 = \lambda_2 W_2$$

Comparison of the three total costs shows which of the priority discipline minimizes the average total cost function of inactivity.

### 3. Fuzzy priority queues

Fuzzy priority queues are described by fuzzy set theory. This paper develops fuzzy priority queuing model in which the input source arrival rate and service rate are uncertain parameters. Approximate methods of extension is propagating fuzziness for continuous valued mapping determined the membership functions for the output variable.

DSW is one of the approximate methods make use of intervals at various  $\alpha$ -cut levels in defining membership functions. It was the full  $\alpha$ -cut intervals in a standard interval analysis. The DSW algorithm greatly simplifies manipulation of the extension principle for continuous valued fuzzy variables, such as fuzzy numbers defined on the real line. It prevent abnormality in the output membership function due to application of the discrimination teaching on the fuzzy variable domain, and it can prevent the widening of the resulting functional value set due to multiple occurrence of variables in the functional expression by conventional interval analysis methods.

#### 3.1. Interval analysis arithmetic

Let  $I_1$  and  $I_2$  be two interval numbers defined by ordered pairs of real numbers with lower and upper bounds.

$$I_1 = [a, b], a \leq b ; I_2 = [c, d], c \leq d.$$

Define a general arithmetic property with the symbol  $*$ , where  $*$  = [+ , - ,  $\times$  ,  $\div$ ] symbolically the operation.

$$I_1 * I_2 = [a, b] * [c, d]$$

represents another interval. The interval calculation depends on the magnitudes and signs of the element a, b, c, d.

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \bullet [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$[a, b] \div [c, d] = [a, b] \bullet \left[ \frac{1}{d}, \frac{1}{c} \right] \text{ provided that } 0 \notin [c, d]$$

$$\alpha [a, b] = \begin{cases} [\alpha a, \alpha b] & \text{for } \alpha > 0 \\ [\alpha b, \alpha a] & \text{for } \alpha < 0 \end{cases}$$

where ac, ad, bc, bd, are arithmetic products and  $\frac{1}{d}, \frac{1}{c}$  are quotients.

### 3.2. DSW algorithm

Any continuous membership function can be represented by a continuous sweep of  $\alpha$ -cut interval from  $\alpha = 0$  to  $\alpha = 1$ . Suppose we have single input mapping given by  $y = f(x)$  that is to be extended for fuzzy sets  $\tilde{B} = f(\tilde{A})$  and we want to decompose  $\tilde{A}$  into the series of  $\alpha$ -cut intervals say  $I_\alpha$ . It uses the full  $\alpha$ -cut intervals in a standard interval analysis. The DSW algorithm [14] consists of the following steps:

1. Selected a  $\alpha$  cut value where  $0 \leq \alpha \leq 1$ .
2. Find the intervals in the input membership functions that correspond to this  $\alpha$ .
3. Using standard binary interval operations, compute the interval for the output membership function for the selected  $\alpha$  cut level.
4. Repeat steps 1 -3 for different values of  $\alpha$  to complete a  $\alpha$  cut representation of the solution.

### 4. Solution procedure

Decisions relating the priority discipline for a queuing system are mainly based for a cost function.

$$C = \sum_{i=1}^p C_i L_i$$

Where  $C_i$  is the unit cost of inactivity for units in class  $i$ ,  $L_i$  is the average length in the system for unit of class  $i$ . Let us consider a queuing model with two unit classes arrive at  $\alpha_1$  of arrivals belong to one of the classes, and  $\alpha_2$  are in the other class. The average arrival rate at the system follows a Poisson process, is approximately known and is given by the triangular fuzzy number  $\tilde{\lambda}$ , the service rate from a single server is the same for both unit classes, follows an exponential pattern and is distributed according to the triangular fuzzy number  $\tilde{\mu}$ , with membership function  $\mu_{\tilde{\lambda}}$ ,  $\mu_{\tilde{\mu}}$  respectively.

$$\mu_{\tilde{\lambda}} = \begin{cases} \frac{\lambda - a}{b - a}, & a \leq \lambda \leq b \\ \frac{c - \lambda}{c - b}, & b \leq \lambda \leq c \\ 0 & \text{else where} \end{cases} \quad \mu_{\tilde{\mu}} = \begin{cases} \frac{\mu - a_1}{b_1 - a_1}, & a_1 \leq \mu \leq b_1 \\ \frac{c_1 - \mu}{c_1 - b_1}, & b_1 \leq \mu \leq c_1 \\ 0, & \text{else where} \end{cases}$$

The possible distribution of unit cost of inactivity for unit in the same class, is established by a triangular fuzzy number  $\tilde{C}_A$ ,  $\tilde{C}_B$  with membership function.

$$\mu_{\tilde{C}_A} = \begin{cases} \frac{C_A - a_2}{b_2 - a_2}, & a_2 \leq C_A \leq b_2 \\ \frac{c_2 - C_A}{c_2 - b_2}, & b_2 \leq C_A \leq c_2 \\ 0, & \text{elsewhere} \end{cases} \quad \mu_{\tilde{C}_B} = \begin{cases} \frac{C_B - a_3}{b_3 - a_3}, & a_3 \leq C_B \leq b_3 \\ \frac{c_3 - C_B}{c_3 - b_3}, & b_3 \leq C_B \leq c_3 \\ 0, & \text{else where} \end{cases}$$

we choose three values of  $\alpha$  viz, 0, 0.5 and 1. For instance when  $\alpha = 0$ , we obtain 4 intervals as follows.

$$\tilde{\lambda}_0 = [a, c] ; \tilde{\mu}_0 = [a_1, c_1] ; \tilde{C}_{A,0} = [a_2, c_2] ; \tilde{C}_{B,0} = [a_3, c_3]$$

Similarly when,  $\alpha = 0.5, 1$ , we obtain 8 intervals and it is denoted by  $\tilde{\lambda}_{.5}, \tilde{\mu}_{.5}, \tilde{C}_{A,.5}, \tilde{C}_{B,.5}, \tilde{\lambda}_1, \tilde{\mu}_1, \tilde{C}_{A,1}, \tilde{C}_{B,1}$ .

The average total cost of inactivity in three situation (i) No priority discipline (ii) Preemptive priority discipline (ii) Non-preemptive priority discipline are calculate for different  $\alpha$  level values. Interval arithmetic is used for computational efficiency.

(i) Average cost of inactivity when there is no priority discipline.

$$\tilde{C}_0 = (\tilde{c}_{1,0} \tilde{\lambda}_{1,0} + \tilde{c}_{2,0} \tilde{\lambda}_{2,0}) \left( \frac{1}{\tilde{\mu}_0 - \tilde{\lambda}_0} \right)$$

$$\tilde{C}_{.5} = (\tilde{c}_{1,.5} \tilde{\lambda}_{1,.5} + \tilde{c}_{2,.5} \tilde{\lambda}_{2,.5}) \left( \frac{1}{\tilde{\mu}_{.5} - \tilde{\lambda}_{.5}} \right)$$

$$\tilde{C}_1 = \frac{(\tilde{c}_{1,1} \tilde{\lambda}_{1,1} + \tilde{c}_{2,1} \tilde{\lambda}_{2,1})}{(\tilde{\mu}_1 - \tilde{\lambda}_1)}$$

(ii) Average total cost of inactivity when there is a preemptive discipline.

$$\tilde{C}_0^1 = \tilde{c}_{1,0} \alpha_1 \tilde{\lambda}_0 \left( \frac{\frac{\tilde{\lambda}_0}{\tilde{\mu}_0^2}}{\left(1 - \frac{\lambda_0}{\mu_0}\right) \left(1 - \alpha_2 \frac{\lambda_0}{\mu_0}\right)} + \frac{1}{\mu_0} \right) + \tilde{c}_{2,0} \alpha_2 \tilde{\lambda}_0 \left( \frac{\frac{\tilde{\lambda}_0}{\tilde{\mu}_0^2}}{\left(1 - \frac{\alpha_2 \lambda_0}{\mu_0}\right)} + \frac{1}{\mu_0} \right)$$

$$\tilde{C}_{.5}^1 = \tilde{c}_{1,.5} \alpha_1 \tilde{\lambda}_{.5} \left( \frac{\frac{\tilde{\lambda}_{.5}}{\tilde{\mu}_{.5}^2}}{\left(1 - \frac{\tilde{\lambda}_{.5}}{\tilde{\mu}_{.5}}\right) \left(1 - \alpha_2 \frac{\lambda_{.5}}{\mu_{.5}}\right)} + \frac{1}{\mu_{.5}} \right) + \tilde{c}_{2,.5} \alpha_2 \tilde{\lambda}_{.5} \left( \frac{\frac{\lambda_{.5}}{\mu_{.5}^2}}{\left(1 - \frac{\alpha_2 \lambda_{.5}}{\mu_{.5}}\right)} + \frac{1}{\mu_{.5}} \right)$$

$$\tilde{C}_1^1 = \tilde{c}_{1,1} \alpha_1 \tilde{\lambda}_1 \left( \frac{\frac{\tilde{\lambda}_1}{\tilde{\mu}_1^2}}{\left(1 - \frac{\tilde{\lambda}_1}{\tilde{\mu}_1}\right) \left(1 - \alpha_2 \frac{\tilde{\lambda}_1}{\tilde{\mu}_1}\right)} + \frac{1}{\tilde{\mu}_1} \right) + \tilde{c}_{2,1} \alpha_2 \tilde{\lambda}_1 \left( \frac{\frac{\tilde{\lambda}_1}{\tilde{\mu}_1^2}}{\left(1 - \frac{\tilde{\lambda}_1}{\tilde{\mu}_1}\right)} + \frac{1}{\tilde{\mu}_1} \right)$$

(iii) Average total cost of inactivity when there is a non-preemptive discipline.

$$\tilde{C}_0^2 = \tilde{c}_{1,0} \alpha_1 \tilde{\lambda}_0 \left( \frac{\frac{1}{\tilde{\mu}_0}}{\left(1 - \frac{\tilde{\lambda}_0}{\tilde{\mu}_0}\right) \left(1 - \alpha_2 \frac{\tilde{\lambda}_0}{\tilde{\mu}_0}\right)} \right) + \tilde{c}_{2,0} \alpha_2 \tilde{\lambda}_0 \left( \frac{\frac{1}{\tilde{\mu}_0}}{\left(1 - \alpha_2 \frac{\tilde{\lambda}_0}{\tilde{\mu}_0}\right)} \right)$$

$$\tilde{C}_{.5}^2 = \tilde{c}_{1,.5} \alpha_1 \tilde{\lambda}_{.5} \left( \frac{\frac{1}{\tilde{\mu}_{.5}}}{\left(1 - \frac{\tilde{\lambda}_{.5}}{\tilde{\mu}_{.5}}\right) \left(1 - \alpha_2 \frac{\tilde{\lambda}_{.5}}{\tilde{\mu}_{.5}}\right)} \right) + \tilde{c}_{2,.5} \alpha_2 \tilde{\lambda}_{.5} \left( \frac{\frac{1}{\tilde{\mu}_{.5}}}{\left(1 - \alpha_2 \frac{\tilde{\lambda}_{.5}}{\tilde{\mu}_{.5}}\right)} \right)$$

$$\tilde{C}_1^2 = \tilde{c}_{1,1} \alpha_1 \tilde{\lambda}_1 \left( \frac{\frac{1}{\tilde{\mu}_1}}{\left(1 - \frac{\tilde{\lambda}_1}{\tilde{\mu}_1}\right) \left(1 - \alpha_2 \frac{\tilde{\lambda}_1}{\tilde{\mu}_1}\right)} \right) + \tilde{c}_{2,1} \alpha_2 \tilde{\lambda}_1 \left( \frac{\frac{1}{\tilde{\mu}_1}}{\left(1 - \alpha_2 \frac{\tilde{\lambda}_1}{\tilde{\mu}_1}\right)} \right)$$

Comparison of the three total costs shows which of the priority discipline is preferable.

## 5. Numerical Example

Consider a centralized parallel processing system in which jobs arrive in two class with utilization of 15% and 85%. Jobs arrive at this system in accordance with a Poisson process and the service times follow an exponential distribution. Both the group arrival rate and service rate are triangular fuzzy numbers represented by  $\tilde{\lambda} = (26, 30, 32)$  and  $\tilde{\mu} = (38, 40, 45)$  per minute, respectively. The possibility distribution of unit cost of inactivity for units of the two classes are triangular fuzzy numbers  $\tilde{C}_A = (15, 20, 22)$  and  $\tilde{C}_B = (2.5, 3, 5)$  respectively. The system manager wants to evaluate the total cost of inactivity when there is no priority discipline, preemptive priority discipline, non-preemptive priority discipline in the queue.

No priority discipline :  $C_0 = (5.915, 40.250)$ ,  $C_{.5} = (5.645, 21.264)$ ,  $C_1 = (16.65, 16.65)$

Preemptive priority discipline:  $\tilde{C}_0^1 = (12.421, 63.506)$ ,  $\tilde{C}_5^1 = (12.427, 37.031)$ ,  $\tilde{C}_1^1 = (26.777, 26.777)$ .

Non Preemptive priority discipline:  $\tilde{C}_0^2 = (8.18, 65.188)$ ,  $\tilde{C}_5^2 = (13.272, 39.71)$ ,  $\tilde{C}_1^2 = (30.01, 30.01)$ . Comparison of the three total costs shows which of the priority disciplines minimizes the average total cost function of inactivity. Even though they are overlapping fuzzy numbers, so minimum average total cost of inactivity is achieved with the non preemptive discipline. The conclusion can therefore be made that the optimum selection of a priority discipline for the fuzzy queuing model that we studied entails establishing a non preemptive priority discipline, in which class A units will be assigned a higher priority.

## 5. Conclusion

Fuzzy set theory has been applied to a number of queuing system to provide broader application in many fields. In this paper measures to apply uncertainty of the initial information when some of the parameters of the models are fuzzy. The method proposed enables reasonable solution to be for each case, with different level of possibility, ranging from the most pessimistic to the most optimistic scenario. This approach provides more information to help design fuzzy priority discipline queuing system.

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