Comparing Arash Model with SBM in DEA

Dariush Khezrimotlagh*, Zahra Mohsenpour and Shaharuddin Salleh

Department of Mathematics, Faculty of Science, UTM, Johor, Malaysia

Abstract

This paper demonstrates the advantages of Arash Method (AM) in comparison with slack based measure (SBM) to benchmark decision making units (DMUs). The paper also suggests the nonlinear AM. A proposition and a numerical example are also illustrated the validity of the statements.

Mathematics Subject Classification: 90

Keywords: Data envelopment analysis, Arash method, Slack based measure, Technical efficiency, Efficiency.

1. Introduction

Arash Method (AM) [1] was recently proposed in data envelopment analysis (DEA) to improve assessing the performance evaluation of decision making units (DMUs) where the weights are not available. It is also able to remove the previous shortcomings in super-efficiency models to arrange technical efficient DMUs where the weights are unknown. This paper after the background of the study in Section 2 illustrates the relation between 0-AM and slack based measure (SBM) in Section 3 and identifies the capabilities of ε-AM in comparison with SBM. The outcomes are also examined with a numerical example and the paper is concluded in Section 4.

2. Background

A DMU is technical efficient if and only if the performances of other DMUs do not show that some of its inputs or outputs can be improved without worsening

* Corresponding author e-mail address: khezrimotlagh@gmail.com, Fax: +60 75537800.
some of its other inputs or outputs. An efficient DMU is a technical efficient DMU which has the best combination of its inputs and outputs in comparison with other technical efficient DMUs, i.e., it has the optimum ratio of its produced output to its used input (output/input) among other technical efficient DMUs. Moreover, data envelopment analysis (DEA) is a nonparametric method in operations research to measure the performance evaluation of homogenous DMUs. One of the most common models in DEA is SBM which was proposed by Tone [3]. It is a non-radial model to calculate mix inefficiencies. However, SBM similar to other conventional DEA models is not able to distinguish between technical efficient DMUs. Therefore, Khezrimotlagh et al. [1] proposed the flexible method called, Arash Method (AM), to identify the efficient DMUs among the technical efficient ones where the prices and weights are not available.

In order to illustrate the models, let us assume that there are \( n \) DMUs (DMU\(_i, i = 1,2,\ldots,n \)) with \( m \) nonnegative inputs \( (x_{ij}, j = 1,2,\ldots,m) \) and \( p \) nonnegative outputs \( (y_{lk}, k = 1,2,\ldots,p) \) for each DMU which at least one of its inputs and one of its outputs are not zero. The \( \varepsilon \)-AM and SBM are as following where DMU\(_l\) \((l = 1,2,\ldots,n)\) is evaluated, \( w_j^- \) and \( w_k^+ \) are the user specified weights obtained through values judgment, \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_m) \), \( \varepsilon_j \geq 0 \), \( s_j^- \)'s and \( s_k^+ \)'s are nonnegative slack, for \( j = 1,2,\ldots,m \) and \( k = 1,2,\ldots,p \).

**\( \varepsilon \)-AM:**

\[
\begin{align*}
\text{max} & \sum_{j=1}^{m} w_j^- s_j^- + \sum_{k=1}^{p} w_k^+ s_k^+, \\
\text{Subject to} & \sum_{i=1}^{n} \lambda_i x_{ij} + s_j^- = x_{ij} + \varepsilon_j/w_j^-, \forall j, \\
& \sum_{i=1}^{n} \lambda_i y_{lk} - s_k^+ = y_{lk}, \forall k, \\
& \lambda_i \geq 0, \forall i, \\
& s_j^- \geq 0, \forall j, \\
& s_k^+ \geq 0, \forall k.
\end{align*}
\]

**Targets:**

\[
\begin{align*}
x_{ij}^- = x_{ij} + \varepsilon_j/w_j^-, \forall j, \\
y_{lk}^- = y_{lk} + s_k^+, \forall k,
\end{align*}
\]

**Score:**

\[
A^* = \frac{\sum_{k=1}^{p} w_k^+ y_{lk}}{\sum_{j=1}^{m} w_j^- x_{ij}^-}
\]

**SBM:**

\[
\begin{align*}
\text{min} & \frac{1 - (1/m) \sum_{j=1}^{m} s_j^- / x_{ij}}{1 + (1/p) \sum_{k=1}^{p} s_k^+ / y_{lk}} \\
\text{Subject to} & \sum_{i=1}^{n} \lambda_i x_{ij} - s_j^- = x_{ij} - \varepsilon_j/w_j^-, \forall j, \\
& \sum_{i=1}^{n} \lambda_i y_{lk} - s_k^+ = y_{lk}, \forall k, \\
& \lambda_i \geq 0, \forall i, \\
& s_j^- \geq 0, \forall j, \\
& s_k^+ \geq 0, \forall k.
\end{align*}
\]

**Targets:**

\[
\begin{align*}
x_{ij}^- = x_{ij} - \varepsilon_j/w_j^-, \forall j, \\
y_{lk}^- = y_{lk} + s_k^+, \forall k,
\end{align*}
\]

**Score:**

\[
\rho^* = \frac{1 - (1/m) \sum_{j=1}^{m} s_j^- / x_{ij}}{1 + (1/p) \sum_{k=1}^{p} s_k^+ / y_{lk}}
\]

Moreover, if the weights \( w_j^- \) and \( w_k^+ \) are unknown they define as following for \( j = 1,2,\ldots,m \) and \( k = 1,2,\ldots,p \), and the \( N_j \) and \( M_k \) can be a nonnegative real number regarding to the goals of each DMU.

\[
\begin{align*}
w_j^- &= \begin{cases} N_j & x_j = 0 \\ 1/x_j & x_j \neq 0 \end{cases} & w_k^+ &= \begin{cases} M_k & y_k = 0 \\ 1/y_k & y_k \neq 0 \end{cases}
\end{align*}
\]
Table 1 illustrates the used notations in the models:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>number of DMUs,</td>
</tr>
<tr>
<td>$m$</td>
<td>number of inputs,</td>
</tr>
<tr>
<td>$i$</td>
<td>index of DMUs,</td>
</tr>
<tr>
<td>$j$</td>
<td>index of inputs,</td>
</tr>
<tr>
<td>$k$</td>
<td>index of outputs,</td>
</tr>
<tr>
<td>$l$</td>
<td>index of specific DMU whose efficiency is being assessed,</td>
</tr>
<tr>
<td>$x_{ij}$</td>
<td>observed amount of input $j$ of DMU$_i$,</td>
</tr>
<tr>
<td>$y_{ik}$</td>
<td>observed amount of output $k$ of DMU$_i$,</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>multipliers used for computing linear combinations of DMUs’ inputs and outputs,</td>
</tr>
<tr>
<td>$s^{-}_{ij}$</td>
<td>non-negative slack or potential reduction of input $j$ of DMU$_i$,</td>
</tr>
<tr>
<td>$s^{+}_{ik}$</td>
<td>non-negative slack or potential increase of output $k$ of DMU$_i$,</td>
</tr>
<tr>
<td>$w^{-}_{ij}$</td>
<td>positive specified weight or price for input $j$ of DMU$_i$,</td>
</tr>
<tr>
<td>$w^{+}_{ik}$</td>
<td>positive specified weight or price for output $k$ of DMU$_i$,</td>
</tr>
<tr>
<td>$\rho^*$</td>
<td>the optimal efficiency score of a DMU by SBM,</td>
</tr>
<tr>
<td>$\lambda^*_i$</td>
<td>optimal multipliers to identify the reference sets for a DMU, $i = 1, 2, ..., n$,</td>
</tr>
<tr>
<td>$s^{-*}_{ij}$</td>
<td>optimal slack to identify an excess utilization of input $j$ of DMU$_i$,</td>
</tr>
<tr>
<td>$s^{+*}_{ik}$</td>
<td>optimal slack to identify a shortage utilization of output $k$ of DMU$_i$,</td>
</tr>
<tr>
<td>$x^*_{ij}$</td>
<td>target of input $j$ of DMU$_i$ after evaluation,</td>
</tr>
<tr>
<td>$y^*_{ik}$</td>
<td>target of output $k$ of DMU$_i$ after evaluation.</td>
</tr>
</tbody>
</table>

### 3. Relations between AM and SBM

In this section, a proposition is proved in order to describe the relation between SBM and 0-AM and next their properties and advantages are discussed with a numerical example.

**Proposition:** The results of SBM and 0-AM are the same where the slacks are equality maximized by both models.

**Proof:** Let $x_j \neq 0$, $\forall j$, and $y_k \neq 0$, $\forall k$, $w^{-}_j$ and $w^{+}_k$ be $1/x_j$ and $1/y_k$, respectively, and $\varepsilon = 0$. Then, the constraints of 0-AM and SBM are exactly the same. Assume that the slacks are equality maximized in both models which yields that the targets are also the same. Now, from the following equations the proposition is proved.

\[
\rho^* = \frac{1 - (1/m) \sum_{j=1}^{m} s^{-*}_{ij}/x_{ij}}{1 + (1/p) \sum_{k=1}^{p} s^{+*}_{ik}/y_{ik}} = \frac{1 - (1/m) \sum_{j=1}^{m} (x_{ij} - x^*_{ij})/x_{ij}}{1 + (1/p) \sum_{k=1}^{p} (y^*_{ik} - y_{ik})/y_{ik}}
\]

\[
= \frac{(1/m) \sum_{j=1}^{m} x^*_{ij}/x_{ij}}{(1/p) \sum_{k=1}^{p} y^*_{ik}/y_{ik}} = \frac{p/m}{\sum_{k=1}^{p} y^*_{ik}/y_{ik} / \sum_{j=1}^{m} x^*_{ij}/x_{ij}} = A^*.
\]
Moreover, from the aim of DMUs the terms $s_j^-/x_{ij}$ or $s_j^+/y_{ik}$, can be eliminated in the objective of SBM where $x_j = 0$ or $y_k = 0$, respectively. In this case, $w_j^-$ or $w_k^+$ in 0-AM can be defined as 0, $m$ should be reduced with $m - 1$ or $s$ with $s - 1$, respectively in both models and the proof is completed.

Now consider 12 DMUs with two inputs and two outputs in Table 2. The results of applying 0-AM and SBM are the same for all DMUs, except DMU 8, as the sixth and seventh columns of Table 2 illustrate them.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Input1</th>
<th>Input2</th>
<th>Output1</th>
<th>Output2</th>
<th>0-AM Score</th>
<th>SBM Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>151</td>
<td>100</td>
<td>90</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
<td>131</td>
<td>150</td>
<td>50</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>160</td>
<td>160</td>
<td>55</td>
<td>0.82868</td>
<td>0.82868</td>
</tr>
<tr>
<td>4</td>
<td>27</td>
<td>168</td>
<td>180</td>
<td>72</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
<td>158</td>
<td>94</td>
<td>66</td>
<td>0.72789</td>
<td>0.72789</td>
</tr>
<tr>
<td>6</td>
<td>55</td>
<td>255</td>
<td>230</td>
<td>90</td>
<td>0.68828</td>
<td>0.68828</td>
</tr>
<tr>
<td>7</td>
<td>33</td>
<td>235</td>
<td>220</td>
<td>88</td>
<td>0.87965</td>
<td>0.87965</td>
</tr>
<tr>
<td>8</td>
<td>31</td>
<td>206</td>
<td>152</td>
<td>80</td>
<td>0.77473</td>
<td>0.77329</td>
</tr>
<tr>
<td>9</td>
<td>30</td>
<td>244</td>
<td>190</td>
<td>100</td>
<td>0.90408</td>
<td>0.90408</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
<td>268</td>
<td>250</td>
<td>100</td>
<td>0.76811</td>
<td>0.76811</td>
</tr>
<tr>
<td>11</td>
<td>53</td>
<td>306</td>
<td>260</td>
<td>147</td>
<td>0.86480</td>
<td>0.86480</td>
</tr>
<tr>
<td>12</td>
<td>38</td>
<td>284</td>
<td>250</td>
<td>120</td>
<td>0.93346</td>
<td>0.93346</td>
</tr>
</tbody>
</table>

Indeed, SBM is a nonlinear programming problem which minimizes a ratio of potential reduction of inputs to potential increase of outputs i.e., it minimizes a fraction, whereas 0-AM is a linear programming problem and maximizes the slacks directly and it does not measure the optimum division of outputs to inputs at the same time. For instance, the optimal slacks of DMU 8 by 0-AM are $s_1^- = 1.20283$, $s_2^- = 0$, $s_1^+ = 80.783019$ and $s_2^+ = 0$, whereas they are $s_1^- = 3.309110$, $s_2^- = 0$, $s_1^+ = 0$ and $s_2^+ = 35.865969$ by SBM. Now the objective of 0-AM for these two set of slacks are 0.570267 and 0.555070, respectively, whereas the objective of SBM are 0.774728 and 0.773286, respectively. Hence, 0-AM selects the first set of slacks and SBM selects the second ones. In other words, $\max(A + B) \not\equiv \min[(1 - A)/(1 + B)]$ i.e., they are not equivalent, because for instance $0.1 + 0.5$ is greater than $0.3 + 0.2$, but $(1 - 0.1)/(1 + 0.5) = 0.6$ is not less than $(1 - 0.3)/(1 + 0.2) = 0.5833$.

Moreover, the efficiency amounts for the proposed targets by SBM and 0-AM for DMU 8, regarding to definition of efficiency, i.e., output/input which is a nonlinear equation, are respectively as following where the input and output values of DMU 8 are considered as measures.

\[\frac{[152/152 + 115.865969/80]}{[27.690890/31 + 206/206]} = 1.293183,\]
\[\frac{[232.783019/152 + 80/80]}{[29.797170/31 + 206/206]} = 1.290775.\]
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This shows that the proposed slacks by SBM for DMU 8 are better than those by 0-AM. In fact, although, SBM does not give the maximum summation of slacks, it may characterize the better efficiency score and targets for a DMU in comparison with 0-AM. In order to improve the objective of $\varepsilon$-AM as a nonlinear equation, from the $\varepsilon$-AM targets i.e., $x_{ij}^\ast = x_{ij} + \varepsilon_j w_j - s_j^\ast, \forall j$, and $y_{lk}^\ast = y_{lk} + s_k^\ast, \forall k$, the $\varepsilon$-AM score, i.e., $A^\ast$, can be rewritten as below:

$$\frac{p/m}{\sum_{k=1}^{p}(y_{lk}^\ast/y_{lk})/\sum_{j=1}^{m}(x_{ij}^*/x_{ij})} = \frac{p \cdot \sum_{j=1}^{m} x_{ij}^*/x_{ij}}{m \cdot \sum_{k=1}^{p} y_{lk}^*/y_{lk}}$$

$$= \frac{p \cdot \sum_{j=1}^{m} (x_{ij} + \varepsilon_j x_{ij} - s_j^\ast)/x_{ij}}{m \cdot \sum_{k=1}^{p} (y_{lk} + s_k^\ast)/y_{lk}}$$

$$= \frac{p \cdot m + \sum_{j=1}^{m} (\varepsilon_j s_j - s_j^\ast)/x_{ij}}{p + \sum_{k=1}^{p} s_k^\ast/y_{lk}}$$

$$= \frac{1 + (1/m) \sum_{j=1}^{m} (\varepsilon_j - s_j^\ast)/x_{ij}}{1 + (1/p) \sum_{k=1}^{p} s_k^\ast/y_{lk}}$$

The last equation defines the nonlinear $\varepsilon$-AM as following, where the slacks and $\lambda_i$’s are optimized.

$$\min \frac{1 + (1/m) \sum_{j=1}^{m} (\varepsilon_j - s_j^\ast)/x_{ij}}{1 + (1/p) \sum_{k=1}^{p} s_k^\ast/y_{lk}},$$

Subject to: the constraint of $\varepsilon$-AM.

Now, SBM is the same as nonlinear 0-AM clearly. Moreover, the optimum of nonlinear $\varepsilon$-AM objective is non-increasing by raising the amount of $\varepsilon > 0$ and the model is still able to arrange technical efficient DMUs, but not as stronger as linear $\varepsilon$-AM. For instance, if DMUs have only one single constant output with more than one input, there may not be any differences between the results of nonlinear $\varepsilon$-AM by increasing the amount of $\varepsilon > 0$, whereas linear $\varepsilon$-AM is able to arrange them well. Indeed, in this case, the results by weighted additive model (ADD), which is a linear programming problem, are more acceptable than the results of SBM [2]. As a result, $\varepsilon$-AM with diversity amount of episilons and selecting different weights can be applied for many purposes in DEA whereas SBM has almost always the 0-AM properties.

It is worth to recognize that, when $\varepsilon > 0$ and $A^\ast < 1$ for a DMU, the linear $\varepsilon$-AM suggests it to change its input and output values to the new $\varepsilon$-AM target and otherwise i.e., when $A^\ast \geq 1$, $\varepsilon$-AM warns that the DMU should not accept the new targets of $\varepsilon$-AM, because it may decrease its efficiency score. However, $A^\ast$ in nonlinear $\varepsilon$-AM is always equal or less than 1 and DMUs are suggested to change their data to the targets of nonlinear $\varepsilon$-AM. For instance, linear 0.01-AM
similar to nonlinear 0.01-AM shows that the technical efficient DMU B by the score 0.9973 has the better combination of inputs and outputs in comparison with technical efficient DMU A with the score 0.9970. This outcome also illustrates the differences between nonlinear ε-AM and SBM where ε > 0 and proves the ε-AM capabilities in comparison with SBM. Moreover, the score of nonlinear ε-AM similar to SBM can be used for calculating mix inefficiencies of DMUs [3].

4. Conclusion

This paper illustrates that with possessing ε-AM, there is no need to use the SBM model. The paper clearly identifies the relation between SBM and 0-AM and improves ε-AM as a good benchmarking tool in DEA.

References


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