Asymptotic Expansion for the Price of a UIP Barrier Option in a Binomial Tree Model

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Abstract

An Up and In Put (UIP) barrier option is a type of derivative contract which activates only when the asset path of the instrument breaches a price level called the barrier \( B \) at least once within its lifespan. We will show in this paper the order of convergence of its discrete version to its continuous counterpart following the methodologies in Escaner and Wee [EW07]. Applying the general method of Joshi [J06], we will show that all three discrete formulas in [LMT03] will be transformed into a single integral. As a consequence of this transformation, we will locate the positions of the parameters, the strike price \( K \) and the barrier level \( B \), relative to the initial value of the underlying instrument \( S_0 \) in the binomial tree.

Keywords: up and in option, asymptotic expansion, Joshi’s general method

1 Introduction

An Up and In Put (UIP) barrier option with strike price \( K \) and barrier level \( B \) is a contract which confers to the owner the right to sell units of the underlying instrument at price \( K \) at maturity date \( T \) if and only if the underlying instrument has breached the barrier level on or before maturity. It belongs to a class of derivative contracts known as Knock In barrier
options where the underlying instrument must traverse the barrier level on or before maturity in order for them to be activated. It becomes beneficial to the subscriber when the pay-off \((K - S_T)\) is greater than zero. UIP and the other barrier options are usually traded in over the counter markets and are interesting since they are cheaper compared to vanilla options.

Below are the discrete pricing formulas developed by Levitan, Mitchell and Taylor\cite{LMT03}:

If \(x \leq m\), then

\[
UIP_0 = e^{-rT} \sum_{j=m}^{\lfloor \frac{n+x}{2} \rfloor} \binom{n}{j-m} p^j (1 - p)^{n-j} \left( K - S_0 u^n d^{n-j} \right).
\] (1)

If \(x > m\) and when \((n + m)\) is odd, then

\[
UIP_0 = e^{-rT} \left[ \sum_{j=m}^{\lfloor \frac{n+x}{2} \rfloor} \binom{n}{j-m} p^j (1 - p)^{n-j} \left( K - S_0 u^n d^{n-j} \right) + \sum_{j=\lceil \frac{n+m}{2} \rceil}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \left( K - S_0 u^n d^{n-j} \right) \right].
\] (2)

If \(x > m\) and when \((n + m)\) is even, then

\[
UIP_0 = e^{-rT} \left[ \sum_{j=m}^{\frac{n+x-1}{2}} \binom{n}{j-m} p^j (1 - p)^{n-j} \left( K - S_0 u^n d^{n-j} \right) + \sum_{j=\frac{n+m}{2}}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \left( K - S_0 u^n d^{n-j} \right) \right].
\] (3)

where
\begin{align*}
m & \quad \text{number of up movements necessary to breach the barrier level } B \\
x & \quad \text{number of up movements necessary to breach the strike level } K \\
p & \quad \text{risk-neutral probability} \\
r & \quad \text{risk-free interest rate} \\
u & \quad \text{up factor} \\
d & \quad \text{down factor} \\
n & \quad \text{time steps at maturity.}
\end{align*}
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The continuous Black-Scholes has always been the standard model in pricing derivative contracts. A less accurate alternative, the discrete Cox-Ross-Rubinstein (CRR) model is known to converge to the Black-Scholes when the time steps tend to infinity [H83]. The rate of convergence of the CRR to the Black-Scholes has been the subject of studies for a number of mathematicians. This is because the order of convergence is CRR’s approximation of the Black-Scholes. Provided that the convergence is good, the time steps of the discrete model need not be infinite. Diener and Diener [DD04] in particular achieved a smooth asymptotic expansion for the CRR model of a DIC barrier option by way of bounded coefficients and showed that the order of convergence is \( \frac{1}{\sqrt{n}} \). Mark Joshi in his paper [J06] developed a general method and showed that the convergence is of order \( \frac{1}{n} \). Escaner and Wee [EW07] incorporated this general method of Joshi in computing the asymptotic expansion for the price of a DIC barrier option in a binomial tree model and attained a smooth expansion of the same order as in [J06]. Both papers, [J06] and [EW07], achieved asymptotic expansions with all three parameters \( S_0, B, \) and \( K \) all lying at the center of the binomial tree.

In this paper we aim to compute the asymptotic expansion of a UIP option since we expect that its expansion will hold for the class of Knock-In Put barrier options and show another location for the parameters \( K \) and \( B \) in the binomial tree with the same expansion.

2 Transformations

Before we can compute the asymptotic expansions of equations (1), (2), and (3), we need to transform them into integral forms. This can be done by using a lemma.

**Lemma 1** Let \( n \) and \( k \) be integers where \( 0 \leq k \leq n \) and \( p \) is the probability of an up movement.

\[
\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} = k \binom{n}{k} \int_0^p y^{k-1} (1-y)^{n-k} dy.
\]  

Using this formula we get their respective integral forms. For equation (1), we have

\[
UIP_0 = Ke^{-rT} \left( \frac{p}{1-p} \right)^\left\lfloor \frac{n+x}{2} \right\rfloor^n \left( m + n - \left\lfloor \frac{n+x}{2} \right\rfloor \right) \times \int_0^p y^{n+m-\left\lfloor \frac{n+x}{2} \right\rfloor - 1} (1-y)^{\left\lfloor \frac{n+x}{2} \right\rfloor - m} dy
\]

\[
- S_0 \left( \frac{q}{1-q} \right)^\left\lfloor \frac{n+x}{2} \right\rfloor^n \left( m + n - \left\lfloor \frac{n+x}{2} \right\rfloor \right)
\]
Similarly, for equation (2), we have

\[
UIP_0 = Ke^{-rT} \left[ \left( \frac{p}{1-p} \right)^{\frac{n+m}{2}} - n \right] \left( n + m - \left[ \frac{n+m}{2} \right] \right) \left( n + m - \left[ \frac{n+m}{2} \right] \right)
\times \int_0^p y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy + \left( \left[ \frac{n+x}{2} \right] \right) \left( \left[ \frac{n+x}{2} \right] \right)
\times \int_0^p y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy - \left( \left[ \frac{n+x}{2} \right] \right) \left( \left[ \frac{n+x}{2} \right] \right)
\times \int_0^p y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy + \left( \left[ \frac{n+x}{2} \right] \right) \left( \left[ \frac{n+x}{2} \right] \right)
\times \int_0^p y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy - \left( \left[ \frac{n+x}{2} \right] \right) \left( \left[ \frac{n+x}{2} \right] \right)
\times \int_0^q y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy \right].
\]

(5)

and for equation (3),

\[
UIP_0 = Ke^{-rT} \left[ \left( \frac{p}{1-p} \right)^{\frac{n+m}{2}} - 1 - n \right] \left( m + n - \left[ \frac{n+m}{2} \right] + 1 \right) \left( m + n - \left[ \frac{n+m}{2} \right] + 1 \right)
\times \int_0^p y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - 1 m dy + \left( \left[ \frac{n+m}{2} \right] \right) \left( \left[ \frac{n+m}{2} \right] \right)
\times \int_0^p y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy - \left( \left[ \frac{n+x}{2} \right] \right) \left( \left[ \frac{n+x}{2} \right] \right)
\times \int_0^p y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy + \left( \left[ \frac{n+x}{2} \right] \right) \left( \left[ \frac{n+x}{2} \right] \right)
\times \int_0^q y^{n+m-\left[ \frac{n+m}{2} \right]} (1-y)^{\left[ \frac{n+m}{2} \right]} - m dy \right].
\]

(6)
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\begin{align*}
\times \int_0^q y^{m+n-\frac{n+x}{2}} (1-y)^{\frac{n+x}{2}-1-m} \, dy + \left( \frac{n+m}{2} \right) \left( \frac{n}{n+m} \right) \\
\times \int_0^q y^{\frac{n+x}{2}-1}(1-y)^{n-\left(\frac{n+x}{2}\right)} \, dy - \left( \frac{n+m}{2} \right) \left( \frac{n}{n+x} \right) \\
\times \int_0^q y^{\left[\frac{n+x}{2}\right]-1}(1-y)^{n-\left[\frac{n+x}{2}\right]} \, dy.
\end{align*}

(7)

We utilize Mark Joshi’s general method \[ J06 \] by equating all the exponents of the integrals and setting \( n = 2N + 1 \). This made all three equations (5), (6), and (7) collapsed into the single formula:

\begin{align*}
UIP_0 &= Ke^{-rT} \left( \frac{p}{1-p} \right) \left[ \frac{n+x}{2} \right]^{-n} \left( m+n-\left\lfloor \frac{n+x}{2} \right\rfloor \right) \left( m+n-\left\lfloor \frac{n+x}{2} \right\rfloor \right) \\
&\times \int_0^p y^{n+m-\left[\frac{n+x}{2}\right]-1}(1-y)^{\left[\frac{n+x}{2}\right]-m} \, dy \\
&- S_0 \left( \frac{q}{1-q} \right) \left[ \frac{n+x}{2} \right]^{-n} \left( m+n-\left\lfloor \frac{n+x}{2} \right\rfloor \right) \\
&\times \left( m+n-\left\lfloor \frac{n+x}{2} \right\rfloor \right) \int_0^q y^{n+m-\left[\frac{n+x}{2}\right]-1}(1-y)^{\left[\frac{n+x}{2}\right]-m} \, dy \\
&= Ke^{-rT} \left( \frac{p}{1-p} \right)^0 \left( N+1 \right) \left( \frac{2N+1}{N+1} \right) \int_0^p y^N (1-y)^N \, dy \\
&- S_0 \left( \frac{q}{1-q} \right)^0 \left( N+1 \right) \left( \frac{2N+1}{N+1} \right) \int_0^q y^N (1-y)^N \, dy \\
UIP_0 &= \left( N+1 \right) \left( \frac{2N+1}{N+1} \right) \left( Ke^{-rT} \int_0^p y^N (1-y)^N \, dy - S_0 \int_0^q y^N (1-y)^N \, dy \right). \quad (8)
\end{align*}

3 Results

Theorem 1 Let the pricing formula of the UIP barrier option be

\begin{align*}
UIP_0 &= \left( N+1 \right) \left( \frac{2N+1}{N+1} \right) \left( Ke^{-rT} \int_0^p y^N (1-y)^N \, dy - S_0 \int_0^q y^N (1-y)^N \, dy \right)
\end{align*}

then its asymptotic expansion is given by

\begin{align*}
UIP_0 = \frac{Ke^{-rT}}{2} - S_0 + \frac{Ke^{-rT}}{2} \text{erf} \left( \frac{r - \sigma^2}{2\sigma \sqrt{T}} \right) - S_0 \text{erf} \left( \frac{r + \sigma^2}{2\sigma \sqrt{T}} \right)
\end{align*}
\[
+ \left( \frac{1}{N} \right)^{\frac{1}{2}} \left[ \frac{K e^{-rT}}{4\sqrt{\pi}} \left( e^{-\frac{e^{2-r^2} + \sigma^2}{4\sigma^2} T} - 1 \right) - \frac{S_0}{4\sqrt{\pi}} \left( e^{-\frac{e^{2+r^2} + \sigma^2}{4\sigma^2} T} - 1 \right) \right] + O \left( \frac{1}{N} \right)
\]

where \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \).

**Proof:** The proof of this theorem is patterned after that of Escaner and Wee [EW07] but contains modifications since we are working with a put option.

Let

\[ D = (N + 1) \left( \frac{2N + 1}{N + 1} \right) \int_0^p y^N (1 - y)^N \, dy \]  

and

\[ L = (N + 1) \left( \frac{2N + 1}{N + 1} \right) \int_0^q y^N (1 - y)^N \, dy. \]

Then equation (8) will become

\[ UIP_0 = Ke^{-rT}(D) - S_0(L). \]  

We focus on computing the asymptotic expansion of \( D \) and apply the same for \( L \). As for \( Ke^{-rT} \) and \( S_0 \), we set them aside since these are constants. Rewriting equation (9) and using repeated integration by parts, changing of variables and integrating limits, we get

\[
D = (N + 1) \left( \frac{2N + 1}{N + 1} \right) \left[ \frac{1}{2} \int_0^1 y^N (1 - y)^N \, dy + \frac{1}{2} \int_0^1 y^N (1 - y)^N \, dy \right] \\
- \int_{1/2}^1 y^N (1 - y)^N \, dy + \int_{1/2}^{s+\frac{1}{2}} y^N (1 - y)^N \, dy \\
= \frac{1}{2} + (N + 1) \left( \frac{2N + 1}{N + 1} \right) \int_{1/2}^{s+\frac{1}{2}} y^N (1 - y)^N \, dy \\
= \frac{1}{2} + 2^{-2N} (N + 1) \left( \frac{2N + 1}{N + 1} \right) \int_0^s \left( 1 - 4y'^2 \right)^N \, dy' \\
= \frac{1}{2} + \left[ 2^{-2N} (N + 1) \left( \frac{2N + 1}{N + 1} \right) \right] \left[ N^{\frac{1}{2}} \int_0^s \left( 1 - 4y'^2 \right)^N \, dy' \right]. \tag{12}
\]

Using Stirling’s Formula and another wave of change of variables, the series expansion of (12) is shown to be:

\[
D = \frac{1}{2} + \left[ \frac{2}{\sqrt{\pi}} + O \left( \frac{1}{N} \right) \right] \left[ \int_{\text{g}(\frac{1}{N})} e^{-w^2} \alpha(0) \, dw \right]
\]
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\[ + \left( \frac{1}{N} \right)^{1/2} \int_0^{\tilde{y}(\tilde{y})} e^{-w^2} \frac{dw}{dw}(0) dw + O \left( \frac{1}{N} \right) \]

\[ = \frac{1}{2} + \frac{2}{\sqrt{\pi}} \int_0^{\tilde{y}(\tilde{y})} e^{-w^2} \alpha(0) dw \]

\[ + \left( \frac{1}{N} \right)^{1/2} \frac{2}{\sqrt{\pi}} \int_0^{\tilde{y}(\tilde{y})} e^{-w^2} \frac{d\alpha}{dw}(0) dw + O \left( \frac{1}{N} \right) \] (13)

where \( \alpha \) is an analytic and smooth function while the function \( \tilde{g} \) is also smooth.

Next we consider the expression \( L = (N + 1) \left( \frac{2N + 1}{N + 1} \right) \int_0^q g^N (1 - p)^N dy \). Using the same method and reasoning that was applied to \( D \), it can be shown that the expansion of \( L \) is:

\[ L = (N + 1) \left( \frac{2N + 1}{N + 1} \right) \int_0^q g^N (1 - p)^N dy \]

\[ = \frac{1}{2} + \frac{2}{\sqrt{\pi}} \int_0^{\tilde{f}(\tilde{x})} e^{-w^2} \alpha(0) dw + \left( \frac{1}{N} \right)^{1/2} \frac{2}{\sqrt{\pi}} \int_0^{\tilde{f}(\tilde{x})} e^{-w^2} \frac{d\alpha}{dw}(0) dw + O \left( \frac{1}{N} \right) \]

\[ + O \left( \frac{1}{N} \right) \] (14)

where again, the function \( \alpha \) is analytic and smooth, and the function \( \tilde{f} \) is also smooth.

Therefore, the series expansion of the price formula of a UIP barrier option as given by equation (11) will be:

\[ UIP_0 = Ke^{-rT} (D) - S_0 (L) \]

\[ = Ke^{-rT} \left[ \frac{1}{2} + \frac{2}{\sqrt{\pi}} \int_0^{\tilde{g}(\tilde{y})} e^{-w^2} \alpha(0) dw + \left( \frac{1}{N} \right)^{1/2} \frac{2}{\sqrt{\pi}} \int_0^{\tilde{g}(\tilde{y})} e^{-w^2} \frac{d\alpha}{dw}(0) dw + O \left( \frac{1}{N} \right) \right] \]

\[ - S_0 \left[ \frac{1}{2} + \frac{2}{\sqrt{\pi}} \int_0^{\tilde{f}(\tilde{y})} e^{-w^2} \alpha(0) dw + \left( \frac{1}{N} \right)^{1/2} \frac{2}{\sqrt{\pi}} \int_0^{\tilde{f}(\tilde{y})} e^{-w^2} \frac{d\alpha}{dw}(0) dw + O \left( \frac{1}{N} \right) \right] \]

\[ = \frac{Ke^{-rT} - S_0}{2} + 2Ke^{-rT} \left[ \alpha(0) \int_0^{\tilde{g}(\tilde{y})} e^{-w^2} dw + \left( \frac{1}{N} \right)^{1/2} \frac{d\alpha}{dw}(0) \int_0^{\tilde{g}(\tilde{y})} e^{-w^2} dw \right] \]

\[ - \frac{2S_0}{\sqrt{\pi}} \left[ \alpha(0) \int_0^{\tilde{f}(\tilde{y})} e^{-w^2} dw + \left( \frac{1}{N} \right)^{1/2} \frac{d\alpha}{dw}(0) \int_0^{\tilde{f}(\tilde{y})} e^{-w^2} dw \right] + O \left( \frac{1}{N} \right) \]

\[ = \frac{Ke^{-rT} - S_0}{2} + \left[ \frac{Ke^{-rT}}{2} \right] \frac{\text{erf} \left( \frac{r - \sigma^2}{2\sigma} \sqrt{T} \right) + \left( \frac{1}{N} \right)^{1/2} \frac{Ke^{-rT}}{4\sqrt{\pi}} \left( e^{-2\sigma^2 r^2 T} - 1 \right) \right] \]

\[ + O \left( \frac{1}{N} \right) \]

\[ = \frac{S_0}{2} \left[ \frac{\text{erf} \left( \frac{r + \sigma^2}{2\sigma} \sqrt{T} \right) - \left( \frac{1}{N} \right)^{1/2} \frac{S_0}{4\sqrt{\pi}} \left( e^{-2\sigma^2 r^2 T} + \frac{1}{2\sigma} - 1 \right) \right} + O \left( \frac{1}{N} \right) \]

where \( \text{erf} \) is the error function.
\[ + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N}\right) \]
\[ = \frac{K e^{-rT} - S_0}{2} + \frac{K e^{-rT}}{2} \text{erf}\left(\frac{r - \frac{\sigma^2}{2}}{2\sigma\sqrt{T}}\right) - \frac{S_0}{2} \text{erf}\left(\frac{r + \frac{\sigma^2}{2}}{2\sigma\sqrt{T}}\right) \]
\[ + \left(\frac{1}{N}\right)^{1/2} \left[ \frac{K e^{-rT} e^{-\frac{r^2 + \frac{\sigma^4}{4} T}{4e^2 T}} - S_0}{4\sqrt{\pi}} e^{-\frac{r^2 + \frac{\sigma^4}{4} T}{4e^2 T} - 1} \right] \]
\[ + O\left(\frac{1}{N}\right). \]

This is the asymptotic expansion that we claimed above.

Next, we check what happened to the parameters under this transformed pricing formula.

**Theorem 2** Let \( x \) be the number of up movements needed to breach the strike price \( K \), \( m \) the number of up movements needed to breach the barrier level \( B \) and \( n \) the number of time steps. Then the price formula given by

\[
UIP_0 = (N + 1) \left( \frac{2N + 1}{N + 1} \right) \left( K e^{-rT} \int_0^p y^N (1 - y)^N dy - S_0 \int_0^q y^N (1 - y)^N dy \right)
\]

has an asymptotic expansion in Theorem 1 if either of the following holds true:

1. If \((n + x)\) is odd, then \( x = m = 0 \);
2. If \((n + x)\) is even, then \( x = m = 1 \).

**Proof:** In applying Joshi’s symmetry method, several equations were formed. The solutions of these equations are: \( m = 0 \), \( m = 1 \), \( m = -1 \), \( x = 0 \), and \( x = 1 \). Since the UIP option has three formulas, we need to check the consistency of these solutions. The solution \( m = -1 \) is then excluded for it violates the conditions of a UIP option. Table 1 summarizes the consistency of the solutions for various cases.

Clearly, from Table 1, only two combinations are consistent. These are the cases when \( m = x = 0 \) and \( m = x = 1 \).

4 Conclusion

The asymptotic expansion of the UIP barrier option in this paper is of order \( \frac{1}{n} \). This result is similar in convergence to that of [EW07] and consistent with Joshi’s general method. However, we have shown that such expansion has another combination for the positions of the parameters \( B \) and \( K \) relative to the binomial tree as shown by theorem 2. This is different from the result obtained in [EW07] and from what was conceived by Joshi in [J06].
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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$(n + x)$ & $x$ & Implied Nature of $n$ & Assumed Nature for $(n + m)$ & Implied Nature of $m$ & Admissible Value for $m$ & Calculated Value of $m$ & Remark \\
\hline
Odd & 0 & Odd & Odd & Even & 0 & 0 & Consistent \\
Odd & 0 & Odd & Odd & Even & 1 & 0 & Inconsistent \\
Odd & 1 & Even & Odd & Odd & 1 & 1/2 & Inconsistent \\
Odd & 1 & Even & Even & Even & 0 & 1/2 & Inconsistent \\
Even & 0 & Even & Odd & Even & 0 & 1/2 & Inconsistent \\
Even & 0 & Even & Even & Even & 0 & 1/2 & Inconsistent \\
Even & 1 & Odd & Odd & Even & 0 & 1 & Inconsistent \\
Even & 1 & Odd & Even & Odd & 1 & 1 & Consistent \\
\hline
\end{tabular}
\caption{Table of Admissible Values for $m$ and $x$.}
\end{table}

References


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