Power Series Solutions for Nonlinear Systems of Partial Differential Equations

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Abstract

In this paper analytical solutions of nonlinear partial differential systems are addressed. The solutions are obtained using the technique of power series to solve linear ordinary differential equations. This method ensures the theoretical exactness of the approximate solution. Several systems are solved using this method and comparisons of the approximate solutions with the exact ones are demonstrated.

Mathematics Subject Classification: 35-04, 35D99

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1. Introduction

It is well known that there are several methods that can be used to find general solutions to linear PDEs. On the contrary, for non-linear PDEs it is well known that there are no generally applicable methods to solve such nonlinear equations. A glance at the literature shows that there are some known methods which have been applied to solve special cases of nonlinear PDEs. For example the split-step method is a computational method that has been used to solve specific equations like nonlinear Schrödinger equation [1, 37].
Nevertheless, some techniques can be used to solve several types of nonlinear equations such as the homotopy principle which is the most powerful method to solve underdetermined equations [9]. In some cases, a PDE can be solved via perturbation analysis in which the solution is considered to be a correction to an equation with a known solution [8, 25]. Alternatively, there are numerical techniques that solve nonlinear PDEs such as the finite difference method [2,19,30] and the finite element methods [27,33,34,36]. Many interesting problems in science and engineering can be solved in this way using computers.

A general approach to solve PDEs uses the symmetry property of differential equations, the continuous infinitesimal transformations of solutions to solutions (Lie theory) [14,26,32,35]. The continuous group theory, Lie algebras and differential geometry are used to understand the structure of linear and nonlinear partial differential equations. Then generating integrable equations to find their Lax pairs, recursion operators, Bäcklund transform and finally finding exact analytic solutions to PDEs [17, 22, 23, 28, 31]. Furthermore, other direct methods were developed to find closed form solutions for nonlinear PDEs such as the Tanh method [18, 24], extended Tanh method [39], Exp-function method [15,16], rational exponential method [38], and others [40-42].


Power series is an old technique for solving linear ordinary differential equations [7,20]. The efficiency of this standard technique in solving linear ODE’s with variable coefficients is well known. An extension known as Frobenius method allows tackling differential equations with coefficients that are not analytic [3]. Recently the method has been used to solve nonlinear ODEs, [5,10-13]. Furthermore, Kurulay and Bayram [21] used power series to solve linear second order PDEs.

In this work we apply the power series method to nonlinear PDEs. Analytical solutions are found by using algebraic series. Manipulation of the equations leads to very convenient recurrence relations that ensure the exactness of the solution as well as the computational efficiency of the method. The method is straightforward and can be programmed using any mathematical package. The efficiency of this method is illustrated through some examples and obtained solutions compared with exact solutions.

The general algebra for solving nonlinear ODEs is explained by considering an analytical function $x = x(\tau)$ defined in $\{ I : 0 \leq \tau \leq 1 \}$. Assume its expansion in power series as
Power series solutions for nonlinear systems

\[ x(\tau) = \sum_{k=0}^{\infty} a_{nk} \tau^k \]  

and for any integer m, \( x(\tau) \) of power m is expressed as

\[ x(\tau)^m = \sum_{k=0}^{\infty} a_{mk} \tau^k \]  

The following relation is an essential condition to be satisfied in order to reveal the desired recurrence relation

\[ x^m(\tau) = x^{m-1}(\tau) x(\tau) \]  

After replacing the series expressions in each factor of equation (3), the following recurrence relation is obtained

\[ a_{mk} = \sum_{p=0}^{k} a_{(m-1)p} a_{1(k-p)} = \sum_{p=0}^{k} a_{1p} a_{(m-1)(k-p)} \]  

To find the series expansion of a product of two functions assumes that the functions \( f(\tau) \) and \( g(\tau) \) are analytic at \( \tau = 0 \) and are defined in \( I \). Let

\[ f(\tau) = \sum_{i=0}^{\infty} a_i \tau^i; \quad g(\tau) = \sum_{j=0}^{\infty} b_j \tau^j \]  

and

\[ h(\tau) = f(\tau) g(\tau) \]  

then the function \( h \) is also analytic at \( \tau = 0 \) and defined in \( I \), therefore the series expansion for \( h(\tau) \) is

\[ h(\tau) = \sum_{i=0}^{\infty} c_i \tau^i \]  

where,

\[ c_i = \sum_{s=0}^{i} a_s b_{i-s} = \sum_{s=0}^{i} a_{i-s} b_s ; i = 0, 1, 2, \ldots \]  

And the series expansion for the \( n^{th} \) derivative \( x(\tau) \), can be written as

\[ x^{(n)}(\tau) = \sum_{k=0}^{\infty} \phi_{nk} a_{(k+n)} \tau^k \]  

where, \( \phi_{ij} = (j+1)(j+2)\ldots\ldots(j+k) \), where \( k \) and \( j \) are positive integers.

For more details on solving nonlinear ODEs, see [31, 32].
To generalize this method for solving nonlinear PDEs, let \( u(x, y) \) be a function of two variables, and suppose that it is analytic in the domain \( G \subseteq \mathbb{R}^2 \) and assume that the point \((x_0, y_0)\) in \( G \). The function \( u(x, y) \) is then represented as

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{ij} (x - x_0)^i (y - y_0)^j
\]  

(10)

To find the representation series for any power of \( u(x, y) \) a condition as in (3) will be applied

\[
u^m(x, y) = u^{m-1}(x, y) u(x, y).
\]  

(11)

If the series expansion of \( u^m(x, y) \) is written as

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{ij}^{(m)} (x - x_0)^i (y - y_0)^j
\]  

(12)

then using the relation in (11), the coefficients of \( u^m \) expressed as

\[
a_{ij}^{(m)} = \sum_{k=0}^{j} \sum_{l=0}^{i} a_{kl}^{(m-1)} a_{ij}^{(1)} a_{kl}^{(1)}
\]  

(13)

A representation for any derivative of \( u \) with respect to \( x \) or \( y \), for any order, and for any power of them, can be found by generalization the equivalent relation for ODEs. Some examples will be used to explain the method.

3. Numerical Examples

To illustrate the technique and exactness of the approximate solution, we now investigate some examples of nonlinear PDEs in detail.

**Example 1.** The nonlinear diffusion equation is considered

\[
\frac{\partial}{\partial x} (u^m u_x) = u_x
\]  

(14)

where \( m \) is a positive integer. Let \( m=2 \), then the equation is

\[
\frac{\partial}{\partial x} (u^2 \frac{\partial u}{\partial x}) = u^2 u_{xx} + 2uu_x^2
\]  

(15)

with initial condition
Power series solutions for nonlinear systems

\[ u(x,0) = \frac{x + h}{2\sqrt{c}} \]  

(16)

where \( c > 0 \), and \( h \) is an arbitrary constant. The exact solution of the given equation is

\[ u(x,t) = \frac{x + h}{2\sqrt{c - t}} \]  

[6].

Assume the solution \( u(x,t) \) as a power series in \( x \) and \( t \),

\[ u(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i t^j \]  

(17)

By differentiating both sides of (17) with respect to \( x \), we will get the expansion series of \( u_x \) and \( u_{xx} \)

\[ u_x = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (i+1)a_{(i+1)j} x^i t^j \]  

(18)

\[ u_{xx} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (i+1)(i+2)a_{(i+2)j} x^i t^j \]  

(19)

The power series of \( u^2 \), \( u^2 u_{xx} \) and \( uu_x \) are obtained by applying (11) and the above series

\[ u^2 = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \sum_{s=0}^{i} \sum_{t=0}^{j} a_s a_{(i-s)(j-t)} \right] x^i t^j \]  

(20)

\[ u^2 u_{xx} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \sum_{q=0}^{i} \sum_{p=0}^{j} (i-p+1)(i-p+2)a_{(i-p+2)(j-q)} \left( \sum_{s=0}^{q} \sum_{t=0}^{p} a_s a_{(p-t)(q-s)} \right) \right] x^i t^j \]  

(21)

\[ uu_x = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left[ \sum_{q=0}^{i} \sum_{p=0}^{j} a_{(i-p)(j-q)} \left( \sum_{s=0}^{q} \sum_{t=0}^{p} (t+1)(p-t+1)a_{(t+1)+a_{(p-t+1)(q-s)}} \right) \right] x^i t^j \]  

(22)

Substitute these series into equation (11), to obtain the recurrence relation:

\[ a_{(j+1)} = \frac{1}{j+1} \left[ \sum_{q=0}^{j} \sum_{p=0}^{j} (i-p+1)(i-p+2)a_{(i-p+2)(j-q)} \left( \sum_{s=0}^{q} \sum_{t=0}^{p} a_s a_{(p-t)(q-s)} \right) \right] + \]

\[ 2 \sum_{q=0}^{j} \sum_{p=0}^{j} a_{(i-p)(j-q)} \left( \sum_{s=0}^{q} \sum_{t=0}^{p} (t+1)(p-t+1)a_{(t+1)+a_{(p-t+1)(q-s)}} \right) \]  

(23)

where
By applying the recurrence relations (23) for several values of \( i \) and \( j \), the polynomial approximation for \( u(x,t) \) is obtained

\[
\tilde{u}(x,t) = \frac{h^2}{2\sqrt{c}} + \frac{h t}{4c^{3/2}} + \frac{5ht^3}{32c^{3/2}} + \frac{x}{2\sqrt{c}} + \frac{tx}{4c^{3/2}} + \frac{3t^2 x}{16c^{3/2}} + \frac{5t^3 x}{32c^{3/2}}
\]  

(24)

Table 1 demonstrates the difference between the approximate solution and the exact one for several values of \( x \) and \( t \) when \( h=1 \) and \( c=10 \).

**Table 1: The difference between exact and approximate solutions for Example 1.**

<table>
<thead>
<tr>
<th>( t )</th>
<th>( u(0,0) - \tilde{u}(0,0) )</th>
<th>( u(0.2,0) - \tilde{u}(0.2,0) )</th>
<th>( u(0.4,0) - \tilde{u}(0.4,0) )</th>
<th>( u(0.6,0) - \tilde{u}(0.6,0) )</th>
<th>( u(0.8,0) - \tilde{u}(0.8,0) )</th>
<th>( u(1.0,0) - \tilde{u}(1.0,0) )</th>
</tr>
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<tbody>
<tr>
<td>0.2</td>
<td>7.04 \times 10^{-9}</td>
<td>8.45 \times 10^{-9}</td>
<td>9.86 \times 10^{-9}</td>
<td>1.13 \times 10^{-8}</td>
<td>1.27 \times 10^{-8}</td>
<td>1.41 \times 10^{-8}</td>
</tr>
<tr>
<td>0.4</td>
<td>1.15 \times 10^{-7}</td>
<td>1.38 \times 10^{-7}</td>
<td>1.61 \times 10^{-7}</td>
<td>1.84 \times 10^{-7}</td>
<td>2.07 \times 10^{-7}</td>
<td>2.30 \times 10^{-7}</td>
</tr>
<tr>
<td>0.6</td>
<td>5.92 \times 10^{-7}</td>
<td>7.11 \times 10^{-7}</td>
<td>8.29 \times 10^{-7}</td>
<td>9.48 \times 10^{-7}</td>
<td>1.07 \times 10^{-6}</td>
<td>1.18 \times 10^{-6}</td>
</tr>
<tr>
<td>0.8</td>
<td>1.91 \times 10^{-6}</td>
<td>2.29 \times 10^{-6}</td>
<td>2.67 \times 10^{-6}</td>
<td>3.05 \times 10^{-6}</td>
<td>3.44 \times 10^{-6}</td>
<td>3.82 \times 10^{-6}</td>
</tr>
<tr>
<td>1.0</td>
<td>4.75 \times 10^{-6}</td>
<td>5.70 \times 10^{-6}</td>
<td>6.65 \times 10^{-6}</td>
<td>7.60 \times 10^{-6}</td>
<td>8.56 \times 10^{-6}</td>
<td>9.50 \times 10^{-6}</td>
</tr>
</tbody>
</table>

**Example 2.** Consider the nonlinear PDE

\[
u_t = u^2 u_{xx} + u^3 + u_x + u
\]

(25)

with initial condition \( u(x,0) = \sin x \).

The exact solution to (25) is \( u(x,t) = e^t \sin(x + t) \).

Assume that

\[
u(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i t^j
\]

(26)

then

\[
u_{xx} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (i+1)(i+2) a_{(i+1)(i+2)} x^i t^j
\]

(27)
and

\[ u^2 = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left[ \sum_{s=0}^{j} \sum_{t=0}^{s} a_{it} a_{i(t-j)s} \right] x^j t^i \]  \hspace{2cm} (28)

Therefore,

\[ u^2 u_{xx} = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left[ \sum_{s=0}^{j} \sum_{t=0}^{s} \left( (i-p+1)(i-p+2) a_{i(i-p+2)(i-j)s} \right) \sum_{q=0}^{p} \sum_{r=0}^{q} a_{ir} a_{i(r-j)s} \right] x^j t^i \]  \hspace{2cm} (29)

and,

\[ u^3 = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left[ \sum_{s=0}^{j} \sum_{t=0}^{s} \left( \sum_{p=0}^{s} \sum_{q=0}^{p} a_{1qp} a_{i(l-q)(i-p)} a_{i(l-t+j)} \right) x^j t^i \right] \]  \hspace{2cm} (30)

Substituting the above series into equation (25), the following recurrence equation is obtained

\[
a_{i(l-1)} = \frac{1}{j+1} \left[ \sum_{q=0}^{p} \sum_{p=0}^{q} (i-p+1)(i-p+2) a_{i(i-p+2)(i-j)s} \left( \sum_{s=0}^{\infty} \sum_{t=0}^{j} a_{ir} a_{i(r-j)s} \right) + \left( \sum_{s=0}^{j} \sum_{t=0}^{j} \sum_{p=0}^{i} \sum_{q=0}^{p} a_{1qp} a_{i(l-q)(i-p)} a_{i(l-t+j)} \right) + (i+1)a_{i(l-1)+1} \right] \]  \hspace{2cm} (31)

The approximate solution is computed for \( i = 0,1,2,3,4 \) and \( j = 0,1,2,3,4 \)

\[
\tilde{u}(x,t) = t + t^2 + \frac{t^3}{3} + x + t x \cdot \frac{t^3}{3} + t x^2 + \frac{t^4}{6} x^2 + t^2 x^2 + \frac{t^3}{6} x^3 + t^3 x^3 + \frac{t^4}{18} x^4 + \frac{t^5}{36} x^4 + \frac{t^6}{24} x^4 + \frac{t^7}{72} x^4
\]  \hspace{2cm} (32)

Table 2 below illustrates the difference between the approximate solution obtained by equation (32) and the exact solution.

<table>
<thead>
<tr>
<th>t</th>
<th>( u(0.1,t)-\tilde{u}(0.1) )</th>
<th>( u(0.2,t)-\tilde{u}(0.2,t) )</th>
<th>( u(0.4,t)-\tilde{u}(0.4,t) )</th>
<th>( u(0.6,t)-\tilde{u}(0.6,t) )</th>
<th>( u(0.8,t)-\tilde{u}(0.8,t) )</th>
<th>( u(1,t)-\tilde{u}(1,t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.41 \times 10^{-12}</td>
<td>7.21 \times 10^{-10}</td>
<td>2.77 \times 10^{-8}</td>
<td>3.68 \times 10^{-7}</td>
<td>2.73 \times 10^{-6}</td>
</tr>
<tr>
<td>0.2</td>
<td>2.35 \times 10^{-11}</td>
<td>2.46 \times 10^{-10}</td>
<td>5.23 \times 10^{-9}</td>
<td>1.34 \times 10^{-7}</td>
<td>1.44 \times 10^{-6}</td>
<td>9.22 \times 10^{-6}</td>
</tr>
<tr>
<td>0.4</td>
<td>1.25 \times 10^{-8}</td>
<td>6.63 \times 10^{-8}</td>
<td>1.28 \times 10^{-7}</td>
<td>4.43 \times 10^{-7}</td>
<td>3.11 \times 10^{-6}</td>
<td>1.82 \times 10^{-5}</td>
</tr>
<tr>
<td>0.6</td>
<td>5.01 \times 10^{-7}</td>
<td>1.90 \times 10^{-6}</td>
<td>3.25 \times 10^{-6}</td>
<td>4.89 \times 10^{-6}</td>
<td>1.03 \times 10^{-5}</td>
<td>3.56 \times 10^{-5}</td>
</tr>
<tr>
<td>0.8</td>
<td>6.93 \times 10^{-6}</td>
<td>2.12 \times 10^{-5}</td>
<td>3.46 \times 10^{-5}</td>
<td>4.73 \times 10^{-5}</td>
<td>6.39 \times 10^{-5}</td>
<td>1.08 \times 10^{-4}</td>
</tr>
<tr>
<td>1.0</td>
<td>5.40 \times 10^{-5}</td>
<td>1.40 \times 10^{-4}</td>
<td>2.21 \times 10^{-4}</td>
<td>2.94 \times 10^{-4}</td>
<td>3.63 \times 10^{-4}</td>
<td>4.59 \times 10^{-4}</td>
</tr>
</tbody>
</table>
Example 3. Consider the nonlinear system
\begin{align*}
  u_x + vu_x + u - 1 &= 0 \\
  v_x + uv_x - v - 1 &= 0
\end{align*}
\tag{33}

Subject to the initial conditions \( u(x,0) = e^x, \quad v(x,0) = e^{-x} \). The exact solution is
\( u(x,t) = e^{x-t}, \quad v(x,t) = e^{t-x} \) \[4\].

In order to solve the given system using the power series method, the solutions \( u \) and \( v \) are considered as
\begin{align*}
  u(x,t) &= \sum_{i=0}^{N} \sum_{j=0}^{N} a_{i,j} x^i t^j \tag{34} \\
  v(x,t) &= \sum_{i=0}^{N} \sum_{j=0}^{N} b_{i,j} x^i t^j \tag{35}
\end{align*}

we use the representation of the solutions in equations (34) and (35) to write the power series expansion of the products \( vu_x \) and \( uv_x \). Then we obtain the recursion formulas
\begin{align*}
  a_{i,j+1} &= \frac{1}{j+1} \left[ \delta_{i,j,0} - a_{i,j} + \sum_{t=0}^{i} (s+1)b_{i-s,j} a_{s+1,j} \right] \tag{36} \\
  b_{i,j+1} &= \frac{1}{j+1} \left[ \delta_{i,j,0} + b_{i,j} + \sum_{t=0}^{i} (s+1)d_{i-s,j} b_{s+1,j} \right] \tag{37}
\end{align*}

where \( \delta_{i,j,0} = \begin{cases} 1 & \text{if } i = j = 0 \\ 0 & \text{o.w} \end{cases} \)

After solving (36), and (37) for \( i=0, \ldots, 3 \) and \( j=0, \ldots, 3 \) we obtain the polynomials
\begin{align*}
  u(x,t) &= 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + x - tx + \frac{t^2 x}{2} + \frac{t^3 x}{6} + \frac{x^2}{2} - \frac{t x^2}{2} \\
  &\quad + \frac{t^2 x^2}{4} + \frac{t^3 x^2}{12} + \frac{x^3}{6} + \frac{t x^3}{12} - \frac{t^3 x^3}{36} \quad \tag{38}
\end{align*}

and,
\begin{align*}
  v(x,t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} - x - tx - \frac{t^2 x}{2} + \frac{t^3 x}{6} + \frac{x^2}{2} + \frac{t x^2}{2} \\
  &\quad + \frac{t^2 x^2}{4} + \frac{t^3 x^2}{12} + \frac{x^3}{6} - \frac{t^2 x^3}{12} - \frac{t^3 x^3}{36} \quad \tag{39}
\end{align*}
The difference between approximate solutions and the exact solutions for equation (33) are shown in tables 3 and 4.

Table 3: The difference between exact and approximate solutions of $u(x,t)$ for Example 3.

| $t$  | $|u(0.1)-\hat{u}(0.1)|$ | $|u(0.2)-\hat{u}(0.2)|$ | $|u(0.4)-\hat{u}(0.4)|$ | $|u(0.6)-\hat{u}(0.6)|$ | $|u(0.8)-\hat{u}(0.8)|$ | $|u(1)-\hat{u}(1)|$ |
|------|------------------|------------------|------------------|------------------|------------------|------------------|
| 0    | 0                | $2.61 \times 10^9$ | $3.42 \times 10^7$ | $6.00 \times 10^6$ | $4.62 \times 10^5$ | $2.26 \times 10^4$ |
| 0.2  | $2.48 \times 10^9$ | $8.94 \times 10^9$ | $2.76 \times 10^7$ | $4.91 \times 10^6$ | $3.78 \times 10^5$ | $1.85 \times 10^4$ |
| 0.4  | $3.10 \times 10^7$ | $3.76 \times 10^7$ | $2.32 \times 10^7$ | $3.46 \times 10^6$ | $3.03 \times 10^5$ | $1.51 \times 10^4$ |
| 0.6  | $5.16 \times 10^6$ | $6.31 \times 10^6$ | $7.52 \times 10^6$ | $6.12 \times 10^5$ | $1.39 \times 10^5$ | $1.10 \times 10^4$ |
| 0.8  | $3.8 \times 10^5$  | $4.62 \times 10^5$ | $5.62 \times 10^5$ | $6.62 \times 10^5$ | $6.34 \times 10^5$ | $1.04 \times 10^4$ |
| 1.0  | $1.8 \times 10^4$  | $2.2 \times 10^4$  | $2.63 \times 10^4$ | $3.19 \times 10^4$ | $3.75 \times 10^4$ | $3.95 \times 10^4$ |

Table 4: The difference between exact and approximate solutions of $v(x,t)$ for Example 3.

| $t$  | $|v(0)-\tilde{v}(0)|$ | $|v(0.2)-\tilde{v}(0.2)|$ | $|v(0.4)-\tilde{v}(0.4)|$ | $|v(0.6)-\tilde{v}(0.6)|$ | $|v(0.8)-\tilde{v}(0.8)|$ | $|v(1)-\tilde{v}(1)|$ |
|------|------------------|------------------|------------------|------------------|------------------|------------------|
| 0    | 0                | $2.48 \times 10^9$ | $3.10 \times 10^7$ | $5.16 \times 10^6$ | $3.78 \times 10^5$ | $1.76 \times 10^4$ |
| 0.2  | $2.60 \times 10^9$ | $8.94 \times 10^9$ | $3.76 \times 10^7$ | $6.31 \times 10^6$ | $4.62 \times 10^5$ | $2.15 \times 10^4$ |
| 0.4  | $3.42 \times 10^7$ | $2.76 \times 10^7$ | $2.32 \times 10^7$ | $7.52 \times 10^6$ | $5.62 \times 10^5$ | $2.63 \times 10^4$ |
| 0.6  | $6.00 \times 10^6$ | $4.91 \times 10^6$ | $3.46 \times 10^6$ | $6.12 \times 10^5$ | $6.62 \times 10^5$ | $3.19 \times 10^4$ |
| 0.8  | $4.61 \times 10^5$ | $3.78 \times 10^5$ | $3.03 \times 10^5$ | $1.38 \times 10^5$ | $6.34 \times 10^5$ | $3.75 \times 10^4$ |
| 1.0  | $2.26 \times 10^4$ | $1.85 \times 10^4$ | $1.51 \times 10^4$ | $1.10 \times 10^4$ | $1.05 \times 10^4$ | $3.95 \times 10^4$ |

Example 4. The coupled Burgers equation

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0,$$
$$v_t - v_{xx} - 2vv_x + (uv)_x = 0$$

with initial conditions $u(x,0) = \sin x$, $v(x,0) = \sin x$. The exact solution of the above system is $u(x,t) = v(x,t) = \sin x e^{lt}$ [29].

In order to solve the given system using the PSM, the solutions $u$ and $v$ is considered as

$$u(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} x^i t^j$$

$$v(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} x^i t^j$$
After substitution the appropriate series in the equation, we get the following recurrence relations

\[
\begin{aligned}
a_{i,j+1} &= \frac{1}{j+1} \left[ (i+1)(i+2)a_{i+2,j} + \sum_{r=0}^{j} \sum_{s=0}^{j} 2(s+1)a_{i-s,j-r}a_{s+j} \\
&+ \sum_{r=0}^{j} \sum_{s=0}^{j} (s+1)b_{i-s,j-r}b_{s+j} - \sum_{r=0}^{j} \sum_{s=0}^{j} (s+1)b_{i-s,j-r}a_{s+j} \right] \\
b_{i,j+1} &= \frac{1}{j+1} \left[ (i+1)(i+2)b_{i+2,j} + \sum_{r=0}^{j} \sum_{s=0}^{j} 2(s+1)b_{i-s,j-r}b_{s+j} \\
&+ \sum_{r=0}^{j} \sum_{s=0}^{j} (s+1)a_{i-s,j-r}b_{s+j} - \sum_{r=0}^{j} \sum_{s=0}^{j} (s+1)b_{i-s,j-r}a_{s+j} \right]
\end{aligned}
\]

(43) (44)

Then the desired coefficients are obtained by repeated application of the recurrence relations. Using Mathematica program to solve the preceding recurrence system, for \(i=0, \ldots, 3\) and \(j=0, \ldots, 3\). We obtain the following polynomials for \(u(x,t)\) and \(v(x,t)\)

\[
\begin{aligned}
\bar{u}(x,t) &= x - tx - \frac{t^3 x}{3} - \frac{t^2 x^3}{6} + \frac{2t^3 x^3}{9} \\
\bar{v}(x,t) &= x - tx - \frac{t^3 x}{3} - \frac{t^2 x^3}{6} + \frac{2t^3 x^3}{9}
\end{aligned}
\]

(45)

Since the exact solutions are the same, then the approximate solutions \(u(x,t)\) and \(v(x,t)\) are also the same. Tables 5 and 6 show the convergence of the approximate solutions to the exact ones for different values of \(x\) and \(t\).

Table 5: The difference between exact and approximate solutions of \(u(x,t)\) for Example 4.

| \(t\) | \(|u(0,1)-\bar{u}(0,1)|\) | \(|u(0.2,1)-\bar{u}(0.2,1)|\) | \(|u(0.4,1)-\bar{u}(0.4,1)|\) | \(|u(0.6,1)-\bar{u}(0.6,1)|\) | \(|u(0.8,1)-\bar{u}(0.8,1)|\) | \(|u(1,1)-\bar{u}(1,1)|\) |
|---|---|---|---|---|---|---|
| 0  | 0  | 1.41 \times 10^{-12} | 7.21 \times 10^{-10} | 2.79 \times 10^{-8} | 3.68 \times 10^{-7} | 2.73 \times 10^{-6} |
| 0.2 | 0  | 1.35 \times 10^{-11} | 6.15 \times 10^{-10} | 2.27 \times 10^{-8} | 3.01 \times 10^{-7} | 2.24 \times 10^{-6} |
| 0.4 | 0  | 3.09 \times 10^{-9} | 6.54 \times 10^{-9} | 2.73 \times 10^{-7} | 2.58 \times 10^{-7} | 1.84 \times 10^{-6} |
| 0.6 | 0  | 7.76 \times 10^{-8} | 1.52 \times 10^{-7} | 2.36 \times 10^{-7} | 4.82 \times 10^{-7} | 1.83 \times 10^{-6} |
| 0.8 | 0  | 7.59 \times 10^{-7} | 1.49 \times 10^{-6} | 2.17 \times 10^{-6} | 2.90 \times 10^{-6} | 4.44 \times 10^{-6} |
| 1.0 | 0  | 4.43 \times 10^{-6} | 8.68 \times 10^{-6} | 1.26 \times 10^{-5} | 1.61 \times 10^{-5} | 1.98 \times 10^{-5} |
Table 6: The difference between exact and approximate solutions of $v(x,t)$ for Example 4.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$v(0,0) - v(0.1)$</th>
<th>$v(0.2) - v(0.2)$</th>
<th>$v(0.4) - v(0.4)$</th>
<th>$v(0.6) - v(0.6)$</th>
<th>$v(0.8) - v(0.8)$</th>
<th>$v(1) - v(1.0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.41 \times 10^{-12}$</td>
<td>$7.21 \times 10^{-10}$</td>
<td>$2.79 \times 10^{-8}$</td>
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<td></td>
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</tr>
</tbody>
</table>

**Conclusion**

The method has been successfully applied directly to some examples of nonlinear PDEs without using linearization, perturbation, or restrictive assumptions. It provides the solution in terms of convergent series with easily computable components and the results have shown remarkable performance. The efficiency of this method has been demonstrated by solving some nonlinear PDEs and some systems of nonlinear PDEs. A comparison of this method with the exact solutions were performed and presented.

**References**


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