On a Fixed Point Iteration Procedure

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Abstract

Let $(X, d)$ be a cone metric space and $T$ be a self-map of $X$. In this paper we investigate the convergence of an iteration procedure involving $T$ to a fixed point of $T$.

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1 Introduction

Let $E$ be a real Banach space. A subset $P \subseteq E$ is called a cone in $E$ if it satisfies the following:

(i) $P$ is closed, nonempty and $P \neq \{0\}$.
(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$.
(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space $E$ can be partially ordered by the cone $P \subseteq E$, by defining; $x \leq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ if $y - x \in \text{int} P$, where $\text{int} P$ denotes the interior of $P$. A cone $P$ is called normal if there exists a constant $k > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$.

In the following we suppose that $E$ is a real Banach space, $P$ is a cone in $E$ and $\leq$ is a partial ordering with respect to $P$.

Definition 1.1 ([1]) Let $X$ be a nonempty set. Assume that the mapping $d : X \times X \to E$ satisfies the following:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.
If \( T \) is a self-map of \( X \), then by \( F(T) \) we mean the set of fixed points of \( T \). Also, \( N_0 \) will denote the set of nonnegative integers, i.e., \( N_0 = \mathbb{N} \cup \{0\} \).

**Lemma 1.2** ([3]) Let \( P \) be a normal cone, and let \( \{a_n\} \) and \( \{b_n\} \) be sequences in \( E \) satisfying the following inequality:

\[
a_{n+1} \leq ha_n + b_n,
\]

where \( h \in (0, 1) \) and \( b_n \to 0 \) as \( n \to \infty \). Then \( \lim_{n} a_n = 0 \).

**Definition 1.3** A self-map \( T \) of \((X,d)\) is called semi-compact if the convergence

\[
\|x_n - Tx_n\| \to 0
\]

implies that there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and \( x^* \in X \) such that \( x_{n_k} \to x^* \).

For some sources on this topics see [1–7].

## 2 Main Result

Let \((X,d)\) be a cone metric space and \( T \) be a self-map of \( X \). Let \( x_0 \) be a point of \( X \), and assume then

\[
x_{n+1} = f(T, x_n)
\]

is an iteration procedure involving \( T \), which yields a sequence \( \{x_n\} \) of points from \( X \). Here we want to investigate the convergence of the iteration procedure

\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) S_n x_n
\]

to a fixed point of \( T \), where

\[
S_n = \frac{1}{n}(I + T + ... + T^{n-1}).
\]

**Definition 2.1** Let \( X \) be a vector space over the field \( F \). Assume that the function \( p : X \to E \) having the properties:

(a) \( p(x, y) \geq 0 \) for all \( x, y \) in \( X \).
(b) \( p(x + y) \leq p(x) + p(y) \) for all \( x, y \) in \( X \).
(c) \( p(\alpha x) = |\alpha| p(x) \) for all \( \alpha \in F \) and \( x \in X \).

Then \( p \) is called a cone seminorm on \( X \). A cone norm is a cone seminorm \( p \) such that

(d) \( x = 0 \) if \( p(x) = 0 \).
We will denote a cone norm by $\| \cdot \|_c$ and $(X, \| \cdot \|_c)$ is called a cone normed space. Also,
\[ d_c(x, y) = \| x - y \|_c \]
defines a cone metric on $X$.

**Theorem 2.2** Let $(X, \| \cdot \|_c)$ be a cone normed space with respect to a normal cone $P$ in the real Banach space $E$, and $T$ be a self-map of $X$ with $F(T) \neq \emptyset$ and
\[ d_c(Tx, q) \leq (1 + \alpha) d_c(x, q) \]
for all $x \in X$ and $q \in F(T)$ where
\[ \sum_{n \in \mathbb{N}_0} \alpha_n < \infty. \]
Suppose that there exists a sequence $\{ \beta_n \} \subset (0, 1]$ such that
\[ \sum_n \frac{1 - \beta_n}{n} < \infty \]
and the sequence $\{ x_n \}$ obtained by the iteration procedure
\[ x_{n+1} = \beta_n x_n + (1 - \beta_n) S_n x_n \]
be bounded where
\[ S_n = \frac{1}{n}(I + T^1 + \ldots + T^{n-1}). \]
Then $\lim d_c(x_n, q)$ exists for all $q \in F(T)$. Moreover, if $T$ is a continuous semi-compact mapping and $d_c(Tx_n, x_n) \to 0$ as $n \to \infty$, then $\{ x_n \}$ convergence to a point of $T$.

**Proof.** Let $q \in F(T)$ and put
\[ \alpha = \sum_n \alpha_n, \]
\[ \gamma_0 = \sup d_c(x_n, q), \]
\[ b_n = d_c(x_n, q) \]
for each $n$. By taking $\alpha_0 = 0$, we get
\[ b_{n+1} = d_c(x_{n+1}, q) \]
\[ = d_c(\beta_n x_n + (1 - \beta_n) S_n x_n, q) \]
\[ \leq \beta_n d_c(x_n, q) + (1 - \beta_n) d_c(S_n x_n, q) \]
\[ = \beta_n b_n + (1 - \beta_n) d_c(S_n x_n, q). \]
But,

\[ d_c(S_n x_n, q) = d_c\left(\frac{1}{n}(x_n + T^1 x_n + \cdots + T^{n-1} x_n), nq\right) \]

\[ \leq \frac{1}{n} \sum_{i=0}^{n-1} d_c(T^ix_n, q) \]

\[ \leq \frac{1}{n} \sum_{i=0}^{n-1} (1 + \alpha_i) d_c(x_n, q) \]

\[ = \frac{1}{n} b_n \sum_{i=0}^{n-1} (1 + \alpha_i) \]

\[ = b_n + \frac{1}{n} \sum_{i=1}^{n-1} \alpha_i. \]

Hence we get

\[ b_{n+1} \leq \beta_n b_n + (1 - \beta_n)(b_n + \frac{1}{n} b_n \sum_{i=1}^{n-1} \alpha_i) \]

\[ = b_n + \frac{1}{n} (1 - \beta_n) \sum_{i=1}^{n-1} \alpha_i b_n \]

\[ \leq b_n + \frac{1}{n} (1 - \beta_n) \alpha b_n \]

\[ \leq b_n + \frac{1}{n} (1 - \beta_n) \alpha \gamma_0. \]

But

\[ \sum_n \frac{1 - \beta_n}{n} < \infty, \]

thus

\[ \lim_{k} \| \sum_{n=1}^{k} (b_{n+1} - b_n) \| \]

exists. But

\[ \sum_{n=1}^{k} (b_{n+1} - b_n) = b_{k+1} - b_1. \]

Hence \( \lim_n \|b_n\| \) exists and so the proof of the first part is complete. Now let \( T \) be continuous semi-compact and \( d_c(T x_n, x_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( T \) is semi-compact, there exists a subsequence \( \{ x_{n_k} \} \) of \( x_n \) and \( q \in X \) such that \( d_c(x_{n_k}, q) \rightarrow 0 \). But \( T \) is continuous, thus

\[ d_c(T x_{n_k}, T q) \rightarrow 0 \]
as $k \to \infty$. Now we have

$$d_c(Tq, q) \leq d_c(Tq, Tx_{n_k}) + d_c(Tx_{n_k}, q) + d_c(q, x_{n_k})$$

which tends to 0 as $k \to \infty$. Hence $Tq = q$. So $q \in F(T)$ and $d_c(x_{n_k}, q) \to 0$. Also, we saw by the first part of the proof, $\lim_n d_c(x_{n_k}, q)$ exists. This implies that $d_c(x_{n_k}, q) \to 0$ and so the proof is complete. $\square$

References


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