Sensitivity of American Option Prices with Different Strikes, Maturities and Volatilities

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Abstract
We compare value functions of the American put options with different strikes, maturities and volatilities based on systematic use of the Dynamic Programming Principle together with the monotonicity in volatility property for the value functions of the American options. Since volatility and strike prices play a vital role in option pricing so these comparisons are also important from practical point of view.

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1 Introduction
We consider a probability space \((\Omega, \mathcal{F}, P)\) and a standard Wiener process \((W_t), 0 \leq t \leq T,\) on it. It is assumed that the time horizon \(T\) is finite and the filtration \(F = (\mathcal{F}_t)0 \leq t \leq T\) is the augmentation of the natural filtration
of \((W_t), 0 \leq t \leq T\). On the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P), 0 \leq t \leq T\), we consider a financial market with two assets \((B_t, S_t), 0 \leq t \leq T\) where \(B_t\) is the value of the unit bank account at time \(t\) and \(S_t\) is the stock value at time \(t\). The evolution of these values obeys the stochastic differential equations

\[
\begin{align*}
\, dB_t &= r(t) \cdot B_t \cdot dt, \quad 0 \leq t \leq T, B_0 = 1 \quad (1.1) \\
\, dS_t &= r(t) \cdot S_t \cdot dt + S_t \cdot \sigma(t, S_t) \cdot dW_t, \quad 0 \leq t \leq T, S_0 > 0, \quad (1.2)
\end{align*}
\]

where the continuous interest rate \(r(t), 0 \leq t \leq T\) satisfies

\[
0 < \underline{r} \leq r(t) \leq \overline{r}, \quad 0 \leq t \leq T, \quad (1.3)
\]

and \(\sigma(t, x)\) is continuous function with respect to pair \((t, x)\) from \([0, T] \times [0, \infty)\) to \((0, \infty)\), such that the diffusion function \(\tilde{\sigma}(t, x) = x \cdot \sigma(t, x)\) is Hölder continuous with exponent \(1/2\) with respect to \(x\) uniformly in \(t\).

\[
| \tilde{\sigma}(t, y) - \tilde{\sigma}(t, x) | \leq \tilde{c} \cdot | y - x |^{1/2}, \quad (1.4)
\]

and \(0 < \underline{\sigma} \leq \sigma(t, x) \leq \overline{\sigma}. \quad (1.5)
\]

We shall state the American put option problem following the general framework of chapter 2 in Karatzas, Shreve [7]. Consider at first the discounted payoff process

\[
X_t = e^{-\int_0^t r(v) dv} (K - S_t)^+, 0 \leq t \leq T, \quad (1.6)
\]

which is obviously bounded continuous process.

Theorem 5.8, chapter 2 of Karatzas, Shreve [7] states that the value process \(V_t\) at time \(t\) equals

\[
V_t = e^{-\int_0^t r(v) dv} \cdot Y_t, \quad (a.s.) \quad (1.7)
\]

where the process \((Y_t, \mathcal{F}_t), 0 \leq t \leq T\) is the so-called Snell envelope of the process \((X_t, \mathcal{F}_t), 0 \leq t \leq T\), that is the minimal supermartingale majorizing the latter process.

According to theorem D.7 of the Appendix D [7] the Snell envelope \((Y_t, \mathcal{F}_t), 0 \leq t \leq T\) has right-continuous paths and for arbitrary \((\mathcal{F}_t)_{0 \leq t \leq T}\)–stopping time \(\tau\) the following equality is valid:

\[
Y_\tau = \text{ess sup}_{\tau \leq \sigma \leq T} E(X_\sigma / \mathcal{F}_\tau), (a.s.) \quad (1.8)
\]

where the essential supremum is taken over all \((\mathcal{F}_t)_{0 \leq t \leq T}\)–stopping times \(\sigma\) such that \(\tau \leq \sigma \leq T\). Moreover as the filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\), is generated by Brownian motion \((W_t), 0 \leq t \leq T\), from Theorem D.13, Appendix D [7] we get that actually the Snell envelope \((Y_t, \mathcal{F}_t), 0 \leq t \leq T\) has almost surely continuous paths.
Let us introduce for any $t, 0 \leq t \leq T$, the stopping time $\tau^*_t$
\begin{equation}
\tau^*_t = \inf\{u \geq t : Y_u = X_u\}, 0 \leq t \leq T. \tag{1.9}
\end{equation}

Theorems D.9 and D.12 of Appendix D [7] assert the fundamental Dynamic Programming Principle for the optimal stopping problem, namely that the stopped supermartingale $(Y_{u \wedge \tau^*_t}, F_u), t \leq u \leq T$, is indeed a martingale and hence (a.s.)
\begin{equation}
E(Y_{u \wedge \tau^*_t} / F_t) = Y_t \tag{1.10}
\end{equation}
for arbitrary $t \in [0, T]$. From the previous considerations we conclude that the value process $(V_t, F_t), 0 \leq t \leq T$, of the American put option is related to the optimal stopping problem of the discounted payoff process $(X_t, F_t), 0 \leq t \leq T$, by the relationship
\begin{equation}
V_t = \text{ess sup}_{t \leq \tau \leq T} E\left(e^{-\int_t^\tau r(s)ds} (K - S_\tau)^+ / F_t\right) \quad \text{(a.s.)} \tag{1.11}
\end{equation}
for arbitrary $t, 0 \leq t \leq T$.

To evaluate the latter expression we have to introduce the family of the solutions of the stochastic differential equation (1.2) with arbitrary initial condition $x \geq 0$ and starting instant $t, 0 \leq t \leq T$,
\begin{equation}
\begin{aligned}
dS_u(t, x) &= r(u) \cdot S_u(t, x)du + S_u(t, x) \cdot \sigma(u, S_u(t, x)) \cdot dW_u, \\
t &\leq u \leq T, x \geq 0, S_t(t, x) = x.
\end{aligned} \tag{1.12}
\end{equation}
Note that by Proposition 2.13, chapter 5 of Karatzas, Shreve [8] thanks to condition (1.4) on the diffusion coefficient $\tilde{\sigma}(t, x)$, there exists a unique strong solution for the latter stochastic differential equation.

Let us introduce now the American put value function
\begin{equation}
v(t, x) = \sup_{t \leq \tau \leq T} E\left[e^{-\int_t^\tau r(s)ds} (K - S_\tau(t, x))^+\right], \tag{1.13}
\end{equation}
the so-called continuation set
\begin{equation}
D^T = \{(t, x) : 0 \leq t < T, x > 0 : v(t, x) > g(x) = (K - x)^+\}
\end{equation}
and the stopping region $C^T = \{(t, x) : 0 \leq t \leq T, x \geq 0 : v(t, x) = g(x)\}$. Define also the optimal exercise boundary $b^T(t), 0 \leq t < T$, as follows
\begin{equation}
b^T(t) = \inf\{x : 0 \leq x \leq K : v(t, x) > g(x)\}, 0 \leq t < T.
\end{equation}
The objective of this paper is to prove a new type of comparison result for the value functions (1.13) of the American put options with different strikes, maturities and volatilities.
The basic idea in our approach consists in combining the Dynamic Programming Principle with the systematic use of presently well known monotonicity in volatility property of the European as well as American options value functions with convex payoffs as originally established by Bergman, Grundy and Wiener [2] and by El-Karoui, Jeanblanc-Picque and Shreve [4] and afterwards generalized and refined by Hobson [5], Janson and Tysk [6] and Ekstrom [3] in case of the local volatility, which is only Hölder(1/2) continuous in \( x \).

### 2 The comparison result for the value functions of the American put options

There exists a well-known relationship between the value process \( V_u, s \leq u \leq T \), and the value function \( v(u, x) \) (see equality (2.12) [9])

\[
V_u = v(u, S_u(s, x)), \quad s \leq u \leq T.
\]

Hence the stopping time \( \tau_s^* \) introduced in (1.9) can be written in the following manner

\[
\tau_s^* = \inf\{u \geq s : v(u, S_u(s, x)) = g(S_u(s, x))\}.
\]

Then the stopped stochastic process

\[
e^{-\int_{\tau_s^*}^{\tau_u} r(v) dv} v(\tau_s^* \wedge u, Q_{\tau_s^* \wedge u}(s, x)), \quad s \leq u \leq T, \quad x \geq 0,
\]

is a martingale on the time interval \([s, T]\).

Using the latter property at \( u = s \) and \( u = t \) we get

\[
v(s, x) = E e^{-\int_{\tau_s^*}^{\tau_u} r(v) dv} v(\tau_s^* \wedge t, S_{\tau_s^* \wedge t}(s, x)), \quad s \leq t \leq T. \tag{2.1}
\]

This equality is a variant of the Dynamic Programming Principle in optimal stopping and will be used in the present article.

Now we are going to state the main result which consists in comparison of the value functions of American puts with different strikes, maturities and volatilities. It is inspired by similar result of Achdou [1], Lemma 3.6, concerning the comparison of the solutions of certain variational inequalities.

For the formulation of this result we need to consider the solution \( \overline{S}_u(s, x) \) of the stochastic differential equation (1.12) with constant volatility \( \overline{\sigma} \)

\[
d\overline{S}_u(s, x) = \overline{S}_u(s, x) \cdot r(u) \cdot du + \overline{S}_u(s, x) \cdot \overline{\sigma} \cdot dW_u, \quad s \leq u \leq T, \quad \overline{S}_s(s, x) = x, \quad x \geq 0,
\]

and to introduce the value function of the corresponding optimal stopping problem for arbitrary time interval \([s, t], s \leq t \leq T\), and arbitrary strike price
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\[ C \]

\[ \overline{v}(s, x) = \sup_{s \leq \tau \leq t} E\left[e^{-\int_s^\tau r(v)\,dv} \cdot (C - \overline{S}_\tau(s, x))^+\right], \quad 0 \leq s \leq t \leq T, x \geq 0. \quad (2.2) \]

The next theorem is important from practical viewpoint too since the volatilities and strikes are major ingredients in option pricing. We can see how American option prices fluctuate when we change maturity, volatility and strike.

**Theorem.** The following relationship is valid between American put values with different strikes, maturities and volatilities

\[ v(s, x) \leq \overline{v}(s, x) + (K - C), \quad 0 \leq s \leq t, x \geq 0, \quad (2.3) \]

where \( C \) is arbitrary constant such that \( 0 \leq C \leq b^T(t) \), and \( b^T(t) \) is the value at instant \( t \) of the optimal exercise boundary \( b^T(u), 0 \leq u < T, \) of the optimal stopping problem (1.13).

**Proof.** Let us start from the Dynamic Programming Equation (see section 3 of [9])

\[ v(s, x) = E\left[e^{-\int_s^\tau r(v)\,dv} \cdot (K - (S_{\tau\wedge t}(s, x))^+) \cdot I(\tau \leq t) + e^{-\int_s^\tau r(v)\,dv} \cdot v(t, S_t(s, x)) \cdot I(\tau > t)\right]. \]

Now suppose \( \tau^*_s > t \) then from the definition of the latter stopping time \( \tau^*_s \), we obtain

\[ v(t, S_t(s, x)) > (K - S_t(s, x))^+, \]

that is \( S_t(s, x) > b^T(t) \), and by the decreasing character of the value function \( v(t, x) \) in \( x \) we get

\[ v(t, S_t(s, x)) \leq v(t, b^T(t)) = (K - b^T(t))^+ \leq K - C, \]

on the event \( (\tau^*_s > t) \).

Therefore

\[ v(s, x) \leq E\left[e^{-\int_s^{\tau^*_s} r(v)\,dv} \cdot (K - S_{\tau^*_s}(s, x))^+ \cdot I(\tau \leq t) + e^{-\int_s^{\tau^*_s} r(v)\,dv} \cdot (K - C) \cdot I(\tau > t)\right]. \quad (2.4) \]

Note that

\[ (K - S_{\tau^*_s}(s, x))^+ \leq (K - C) + (C - S_{\tau^*_s}(s, x))^+, \]

then the following estimate follows from the previous inequality

\[ v(s, x) \leq E\left[e^{-\int_s^{\tau^*_s} r(v)\,dv} \cdot (C - S_{\tau^*_s}(s, x))^+ \cdot I(\tau \leq t) + (K - C) \right] \leq \overline{v}(s, x) + (K - C), \]

where

\[ \overline{v}(s, x) = \sup_{s \leq \tau \leq t} E\left[e^{-\int_s^\tau r(v)\,dv} \cdot (C - S_\tau(s, x))^+\right]. \]
By the monotonicity in volatility property of the American option value function we obtain

\[ \tilde{v}(s, x) \leq \pi(s, x), 0 \leq s \leq t \leq T, x \geq 0, \]

and from the previous inequality we arrive to the estimate (2.3), i.e. we have proved the theorem.

\[ \square \]

References


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