Interpolation of a Rectangular Data Array by Powers of Bilinear Expressions

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Abstract
This paper illustrates interpolation of the four-point rectangular data array by powers of bilinear expressions. The new method can be used in two ways. When it applies, it offers the prospect of more accurate interpolating equations.

Mathematics Subject Classification: 65D05, 65D07, 65D17

Keywords: two-dimensional interpolation, experimental design

1. Introduction
The four-point data array, denoted ACIG in Fig. 1, is usually represented by the bilinear equation. The array can also be represented by quadratic and cubic equations, by exponential equations, and by trigonometric equations [1-5]. These alternatives offer the potential for better accuracy without greater expense. A bilinear expression, Eq. (1), is applied in sections 2 and 3. It is written for the four-point rectangle in Fig. 1. The figure follows the references below.

\[ R = \frac{(A + C + G + I)}{4} + \frac{(I + C - G - A)(x/4)}{} + \frac{(G + I - A - C)(y/4)}{} \]  

(1)
2. First application of roots methodology

The data are located at vertices A, C, G, I in Fig. 1. These letters also represent measurements at the corresponding vertices. The first method for the four-point rectangle is based on Eq. (2). The letter N represents the “degree of the data.” This term is descriptive but N is primarily an artifact that is used for deriving interpolation equations. If Eq. (2) has a real, nonzero root, the four-point rectangle can be interpolated by a power of Eq. (1).

\[ A^{(1/N)} + I^{(1/N)} = C^{(1/N)} + G^{(1/N)} \]  
(2)

Let Eq. (2) be rewritten as Eq. (3). There are 24 permutations of [A,C,G,I]. Each member of a particular permutation is substituted as P1,P2,P3,P4, respectively, into Eq. (3). This operation is effected for all 24 permutations. Suppose Eq. (3), in any of its 24 forms, has a root for N. Use it to find the Nth roots of the data.

\[(P_1^{(1/N)} + P_2^{(1/N)} - P_3^{(1/N)})^N - P_4 = 0\]  
(3)

Now substitute the Nth roots of the original data as new A,C,G,I, respectively, into the bilinear expression. Attach N as the exponent to the expression. The result is an equation for the four-point rectangle in Fig. 1. For example, suppose A=1, C=9, G=49, I=81. At least one form of Eq. (3) renders N=2 so the equation interpolating the rectangle is Eq. (4). The letter R represents an interpolated number in the –1 .. 1 coordinate system.

\[ R = (5 + x + 3y)^2 \]  
(4)

The “degree of the data”, N, can be positive or negative. In most problems, N is a fraction lying within the range –1000 .. –0.001 or 0.001 .. 1000. More examples of this approach are found in Ref. [5].

3. Second application of roots methodology

This method is also easy to apply. Rewrite Eq. (3) as Eq. (5). In the second method, the degree of the data, N, is assigned. It can be positive or negative but not zero. Equation (5) now represents a new set of 24 equations. The unknown to be obtained from the new set of 24 equations is the value of T. If the assigned exponent N fails to render a real number for T, change N and try again.
Interpolation of a rectangular data array

\[(P_1+T)^{(1/N)} + (P_2+T)^{(1/N)} - (P_3+T)^{(1/N)} = 0\]  \hspace{1cm} (5)

If \(T\) can be evaluated, add it to each of the original data so that they become \(A+T\), \(C+T\), \(G+T\), and \(I+T\), respectively. Then the \(N^{th}\) root of each sum is found: \((A+T)^{(1/N)}\), \((C+T)^{(1/N)}\), \((G+T)^{(1/N)}\), and \((I+T)^{(1/N)}\). These \(N^{th}\) roots are substituted into right hand side of Eq. (6) as new \(A,C,G,I\), respectively. The expression is raised to the \(N^{th}\) power and the value of \(T\) is subtracted from it. The result is a new equation for the four data in Fig. 1. When this method applies, it yields equations that are invariant under rotation and translation of the data.

\[R = (A + C + G + I)/4 + (I + C - G - A)(x/4) + (G + I - A - C)(y/4) + (I + A - C - G)(xy/4)\]  \hspace{1cm} (6)

For example, let \(A=1, C=3, G=7, I=9\). Let it be desired to represent them by an expression of degree two so that \(N=2\). Application of the method, as represented by Eqs. (5) and (6), renders \(T=6.138\), nearly. The interpolation equation for the four data is Eq. (7). It has the desired degree, \(N=2\), and it reproduces the original data. The true center point response at \((x,y)=(0,0)\) is \(E=5^{3}=125\). Equation (7) yields \(E=177.6\), nearly. That is a better approximation than rendered by the bilinear equation \((E=275)\) even though the degree of the data is wrong.

\[R = (13.56 + 2.878x + 9.343y + 1.336xy)^2 - 6.138\]  \hspace{1cm} (7)

If \(N\) is assigned as \(N=3\), the interpolation equation is exact and \(E=125\). If \(N=4\), the new interpolation equation is Eq. (8). It also has the wrong degree. It renders \(E=119.9\). Both wrong \(N\) values render better estimates of \(E\) than Eq. (6).

\[R = (3.333 + 0.4439x + 1.425y)^4 - 3.594\]  \hspace{1cm} (8)

Let the trial data be \([1,10,7,5]\) as \([A,C,G,I]\), respectively. They can be represented by a quadratic equation, Eq. (9a). The italic letter \(I\) is the imaginary unit. Note that \(I\) can be removed from the expression in parentheses and \(I^2\) squared is \(-1\). Equation (9a) expands to Eq. (9b). If new data are \([1,10,6,11]\) as \([A,C,G,I]\), respectively, the interpolation equation is Eq. (10a) or Eq. (10b).

\[R = (3.10I - 0.282Ix + 0.433Ixy)^2 + 15.7\]  \hspace{1cm} (9a)

\[R = 6.03 + 1.75x - 2.75xy - 0.0796x^2 + 0.250x^2y - 0.196x^2y^2\]  \hspace{1cm} (9b)

\[R = (1.62I - 1.08Ix - 4.63Iy)^2 + 11.0\]  \hspace{1cm} (10a)
If the data are [6,3,1,14] as [A,C,G,I], respectively, and if N=2, Eq. (11) results. If the data are [6,9,3,2] as [A,C,G,I], respectively, the equation is Eq. (12). Such equations should be questioned if they generate spurious ridges or troughs within rectangle ACIG in Fig. 1. The flexibility of choosing N is not available in Eq. (6). Laboratory results are used to suggest the choices for N.

\begin{align*}
R &= (1.83 + 0.685x + 1.10xy)^2 + 0.998 \tag{11} \\
R &= (1.58 - 0.791y - 0.316xy)^2 + 1.78 \tag{12}
\end{align*}

4. Discussion

The roots method, as based on Eq. (5), can answer interesting questions. Suppose the data at A,C,G,I in Fig. 1 are 2,8,128,512, respectively. They are 2,1,2,2, respectively. They have an exponential character. What equation can estimate the center point of the rectangle, E=32, with an accuracy of 0.1%? The answer is a bilinear expression with the exponent N=500. With coefficients rounded, it is Eq. (13). It renders 32.027, nearly, at (x,y)=(0,0). The equation is unusual but it satisfies the requirements of the problem. A closer approach to E=32 is rendered by N=1000. Calculations for roots procedures may require extended numerical precision. The coefficients in Eq. (13) and other equations are rounded.

\begin{align*}
R &= (1.006955 + 0.001402x + 0.004197y)^{500} + 0.03113 \tag{13}
\end{align*}

Roots methods should reproduce the original data. When they apply, they represent opportunities for better accuracy at no increase in cost. This advantage is suggested by Table 1. Sometimes Eq. (5) has more than one root. That is more likely when N is negative. The user then chooses the best surface for his purpose. Modified roots methods also apply to four-point diamond arrays [6,7].

References


![Fig. 1. The four-point rectangle](image)

Table 1. Approximate sums of squares of deviations by two roots methods. The data at A,C,G,I are generated by applying the functions to 1,3,7,9 at vertices A,C,G,I, respectively, in Fig. 1. Reference surfaces are generated from $M=(5+x+3y)$. The assignment $N=2$ is arbitrary.

<table>
<thead>
<tr>
<th>Function</th>
<th>Method 1</th>
<th>Method 2, $N=2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>tan(7M°)</td>
<td>0.00845</td>
<td>0.0116</td>
</tr>
<tr>
<td>sinh(M/4)</td>
<td>0.0194</td>
<td>0.0326</td>
</tr>
<tr>
<td>cosh(M/4)</td>
<td>0.0320</td>
<td>0.0324</td>
</tr>
<tr>
<td>ln(M!)+1</td>
<td>0.0542</td>
<td>0.0670</td>
</tr>
<tr>
<td>M/sin(10M°)</td>
<td>0.0464</td>
<td>0.0174</td>
</tr>
<tr>
<td>M²</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M²+100</td>
<td>19.4</td>
<td>0</td>
</tr>
</tbody>
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Received: May, 2012