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Multi Stage Optimal Mix in the Interconnection of Drinking Water Sources

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Abstract

The use of integrated aqueduct systems is quite common in the provision of drinking water. In fact, it enables the system to handle crisis in the provision of the service caused, for example, by pollution emergencies or peaks in day demand curves.

The Italian Local Water authorities (ATOs) assign a concession contract to a private provider that has the right to produce, operate and manage water utilities for a certain period of time (usually 30 years). The tariffs, set by the ATO according to the Government Decree n. 1/8/1996, can be revised after a period lasting three years. Within each single year some technical and economical data can be considered as fixed, such as the volume of water produced, the operating costs and the tariffs. Therefore the provider's profit can be considered as constant over a single year though it varies on a yearly basis. That's why a deterministic approach can be properly adopted over a three-year period. We formulate and solve an optimal control model in order to determine the optimal abstraction policy for a provider of water services who has invested in the interconnection of two different sources and wants to maximize his profit and to minimize the environmental costs related to water abstraction.

Mathematics Subject Classification: 49N90 , 93C95

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1 Introduction

Urban water problems, and in particular the supply of drinking water, are mounting all over the globe. Widespread mismanagement of water resources, growing competition for the use of fresh water, degraded sources by pollutants or unpredictable effects of climate change increase the negative consequences of these problems. The provision of drinking water on the one hand and its conservation on the other are, therefore, key issues worldwide. The ability of a system to supply consumers requirements under different operating conditions has been widely investigated in the literature by engineers and mathematicians. Starting from the seminal works of [17], [8], [11] and [10], a variety of algorithms for water distribution networks have been developed over recent years to take into account reliability aspects arising from the mechanical failure of the system components and nodal demand variation.

The main challenges for water service providers are to supply high quality water in sufficient quantities at affordable costs whilst maintaining the various ecosystems and better matching water demand with resource availability. Although drinking water supply management is typically modelled as a single source serving a group of consumers, resource providers must often decide to manage multiple sources simultaneously. It might be very difficult and costly to guarantee the reliability of the supply system and continuity in the service provision by using a single source. Consequently the problem might be better set as one of a single demand supplied by multiple sources. Therefore, when more than one resource is available, optimal management involves the conjunctive use of different water sources (see for example [4]). Economic models of conjunctive use consider at least two sources, one of which is flow and the other is stock. This issue has been widely investigated in agricultural economics ([1], [24], [13], [16], [28], [20], [7], [6]). In particular the conjunctive use of multiple sources protects users against uncertainty in provision, (see, among others, [28], [30], [26], [27], [23] and [6]).

Water scarcity may become greater in one source than the another as well as extraction from one source may become more costly than the other. The joint use of two or more sources can therefore lead to a cheaper supply than that gained by their independent use.

Technological innovations lead to the construction of water utilities characterized by a high operational flexibility and high irreversible sunk costs. It is quite common today to design integrated aqueduct systems (namely vertically integrated systems with several interconnections between the network infrastructures). The interconnection and integration between supply sources, in fact, enables the system to handle crisis in the provision of the service caused, for example, by pollution emergencies or peaks in day demand curves. In this paper, we formulate and solve an optimal control model in order to determine the optimal abstraction policy for a provider of water services who has invested in the interconnection of two different water sources (e.g. groundwater versus river abstraction). The interconnection of water abstraction plants gives, *de facto*, the provider the option to strategically decide the optimal mix of different water sources to be used in supplying water to a community. Our aim is to show that this operational and technical flexibility is economically relevant if optimally exercised.

In Italy, the Law n. 36/1994 started the reform of the water service sector and established a net separation between water resource planning and the operation of water utilities. The resource planning is assigned to the local water authority (ATO) which, in turn, assigns the operation to a private provider which will be selected via an auction mechanism. The ATO assigns a concession contract to a private provider that has the right to produce, operate and manage water utilities for a certain period of time (usually 30 years). Moreover the ATO sets the price (i.e. the tariff) cap for the water utilities and draws up the "Piano d'Ambito" (usually a 30 year plan which includes the timing and level of infrastructure investments), and ensures that the provider fulfils the contract requirements. The Government Decree n. 152/2006, which substantially maintains the articles and principles set by the Law n. 36/1994, defines the pricing mechanism. The tariffs are in fact determined on the basis of Presidential Decree 1/8/1996 that introduces a price cap regulation which guarantees at the same time ex-post full recovery of the service costs and an adequate capital rate of return (see [15]).

The law allows the provider to ask for a revision of the tariffs set by the ATO after a three-year period (see Presidential Decree 1/8/1996, art.8) which can be considered as fixed during this time. Within each single year some technical and economical data can be considered as fixed, such as the volume of water produced, the operating costs and the tariffs. Therefore the provider's profit can be considered as constant over a single year though it varies on a yearly basis. That's why a deterministic approach can be properly adopted over a three-year period since afterwards economic and technical input data may change.

The paper is organized as follows. In Section 2 we introduce the variables which describe the water system and we define the optimal control problem of the water provider. In Section 3 we characterize the optimal extraction policy through the necessary optimality conditions. A particular form of the environmental costs is analyzed and in this case sufficiency results can be stated and permit to obtain the optimal solution. Furthermore, we validate such solution via a numerical simulation. A more extensive analysis is done in Section 4, where, in analogy with the cited literature, we tackle the problem in the case of environmental costs not depending on the extraction rate. A numerical simulation is conducted also in this case. Section 5 concludes the paper with some ideas for the future research.

2 The model

In the intent of planning the water extraction over the next three years, let us consider the interval $[0, T_3]$, split into the following three subintervals $I_1 = [0, T_1]$, $I_2 = [T_1, T_2]$, $I_3 = [T_2, T_3]$, with $T_1, T_2, T_3 > 0$.

The analysis of a finite time period is the novelty of our paper, in fact the models in the existing literature consider infinite time horizon and look for steady state solutions, as for example in [18]. When referring to the Italian setting it is crucial to consider a three periods finite horizon, because of the Italian water service regulation. Furthermore, studying each single sub-period would be limiting and only the formulation of a multi-period optimal control problem can guarantee the optimal strategy for the water provider.

Let us denote by $x(t)$ the state function which represents the volume of groundwater at time t (measured in m^3), and let $x_0 > 0$ be the initial water volume. Let us denote by $u(t)$ the control function which gives the extraction rate at a time t (measured in $m^3/\sec$) and let $\overline{u}$ be the maximum extraction rate ($m^3/\sec$) set by locals regulators.

We assume, in line with [19], that the groundwater stock has a fixed and constant recharge rate R, ($m^3/\sec$) and furthermore that the maximum extraction rate is less than the recharge rate, i.e.

$$\bar{u} < R. \tag{1}$$

The latter is quite a strong assumption, nevertheless it is necessary in order to limit the environmental impact and to guarantee a sustainable development of the water resource.

From the definition of the state and control functions it directly follows that the water evolution may be described by the following differential equation

$$\dot{x}(t) = R - u(t), \quad t \in [0, T_3]. \tag{2}$$

In order to guarantee sustainability, we also require that the water stock at the end of the three years is at least greater than the initial one. We can formalize such requirement by mean of the following constraint on the final value of the state function

$$x(T_3) \geq x_0. \tag{3}$$

At any given sub-period $i \in \{1, 2, 3\}$, let us denote by P_{gi} the constant unit profit obtained from the tariff revenue minus the operating costs. We make the following assumptions on the periodical profits

$$P_{g1} < P_{g2} < P_{g3}.$$

Such hypotheses can be explained by the fact that tariffs (revenues) are bound to a price cap mechanism and therefore they can increase over time according to the following rule

$$T_n = T_{n-1}(1 + k + \pi),$$

where T_n is the tariff at year n, T_{n-1} is the tariff at year $n-1$, k is the price cap and π is the planned inflation rate as estimated by the Italian Government. Average revenues per cubic meter can be determined by a statistical analysis performed over a distribution whose parameters are estimated on the basis of the average tariffs paid by users for the provision of drinking water.

As a matter of fact profits could decrease in case of high operating costs. Groundwater operating costs are mainly due to the expenditure on energy necessary to pump the water. So that, the deeper the groundwater, the higher the related costs. Nevertheless here we assume that planned tariffs increments are greater than the operating costs increments, for each period.

We assume that profits increase over time due to the revenues increase which counterbalances the potential operating costs increase. This is a fair assumption because when the service can apply a water price that is above the financial equilibrium price, this constitutes an endowment that, according to the Italian regulation, is intended for self-financing and investing purposes. In particular it is worth noting that the Italian regulation guarantees a return (in percentage terms) on the investments made by the provider and the full cost recovery of the operating costs (see Presidential Decree n. 1/8/1996 and and Government Decree n. 152/2006). The provider is subject to a price cap regulation but it is also guaranteed at time t to recover from the costs of service paid at time $n-1$.

Let $C_e(x(t), u(t))$ be the environmental unit costs function due to the groundwater extraction. Environmental costs coincide with the opportunity cost of removing one unit of water from the aquifer and are associated with the use of water beyond its original designated purpose, to the buffer role of groundwater ([24], [28], [25]) and to common property situations (see among others [16] and [19])

The above literature follows the common empirical formulation which assume the environmental unit costs to be independent of extraction rate and to be decreasing and convex w.r.t. the water volume. In other words, the common formulation consider linear unit costs in the extraction rate. Our model shares the realistic assumption on the monotonicity of the unit costs w.r.t. the variable $x(t)$ and furthermore it considers the possibility that the environmental unit costs also depend on the extraction rate itself. We assume they are increasing and convex w.r.t. the variable $u(t)$. Under the hypotheses of smoothness the former assumptions can be expressed as follows

$$\frac{\partial C_e(x,u)}{\partial x} < 0, \ \frac{\partial^2 C_e(x,u)}{\partial x^2} > 0 \tag{4}$$

and

$$\frac{\partial C_e(x,u)}{\partial u} > 0, \ \frac{\partial^2 C_e(x,u)}{\partial u^2} \geq 0. \tag{5}$$

Assuming that the total quantity of water the manager has to supply at each period is equal to one, from the definition of $u(t)$ it follows that $1-u(t)$ is the surface water extraction rate. In analogy to what defined for groundwater, let P_{si} be the unit profit at period i deriving from surface water abstraction, subject to the following assumptions

$$P_{s1} < P_{s2} < P_{s3}.$$

We further assume that

$$P_{si} < P_{gi}, \quad i \in \{1,2,3\}, \tag{6}$$

according to the fact that operating costs related to river abstraction are usually greater than the groundwater ones, because of the severe sanitization and purification processes that surface water has to undergo with respect to groundwater. This costs turn out to be very expensive in case of a polluted river. On the contrary, aquifer water is typically of good quality because of natural purification processes and groundwater is less susceptible to pollution, see [29], [14], [4].

The objective of the water service provider is to maximize the discounted profits coming from the extraction of the water along the three years. We recall that the groundwater profits are

$$P_{gi}u(t) - C_e(x(t),u(t)), \ i \in \{1,2,3\},$$

whereas profits deriving from surface water extraction are

$$P_{si}(1-u(t)), \ i \in \{1,2,3\}.$$

Observe that, as in [28], we assume, without loss of generality, that supply of surface water is costless from an environmental point of view.

The objective function which describes the total profits over the three periods is

$$\begin{aligned} J(u) = \quad & \int_0^{T_1} e^{-rt}\left(P_{g1}u(t) + P_{s1}(1-u(t)) - C_e(x(t),u(t))\ u(t)\right)\, dt + \\ + \quad & \int_{T_1}^{T_2} e^{-rt}\left(P_{g2}u(t) + P_{s2}(1-u(t)) - C_e(x(t),u(t))\ u(t)\right)\, dt + \\ + \quad & \int_{T_2}^{T_3} e^{-rt}\left(P_{g3}u(t) + P_{s3}(1-u(t)) - C_e(x(t),u(t))\ u(t)\right)\, dt, \end{aligned}$$

where r is the constant discount rate.

After denoting by $\overline{P}_i = P_{gi} - P_{si},\ i \in \{1,2,3\}$, the water manager has to solve the following optimal control problem

$$
\begin{aligned}
\max_u J(u) = & \int_0^{T_1} e^{-rt}\left(\bar{P}_1 u(t) + P_{s1} - C_e(x(t),u(t))\ u(t)\right)dt + \\
+ & \int_{T_1}^{T_2} e^{-rt}\left(\bar{P}_2 u(t) + P_{s2} - C_e(x(t),u(t))\ u(t)\right)dt + \\
+ & \int_{T_2}^{T_3} e^{-rt}\left(\bar{P}_3 u(t) + P_{s3} - C_e(x(t),u(t))\ u(t)\right)dt \\
\text{s. t.} \quad & \dot{x}(t) = R - u(t), \\
& x(0) = x_0, \\
& x(T_3) \geq x_0, \\
& u(t) \in [0,\overline{u}], \quad \bar{u} < R.
\end{aligned}
$$

Let us observe that $\overline{P}_i = P_{gi} - P_{si}$ is generally positive because of (6).

3 Optimal extraction rate

The water management problem can be structured as a three-stage optimal control problem in finite horizon. In [9] optimality conditions are given for a similar problem in infinite horizon; a careful treatment of fixed endpoint problems can be found in [12, pp. 147-153] and in [2] a proof of the Maximum Principle for Two-Stage Optimal Control is given considering a two period interval and a fixed endpoint. Here we consider a three-period interval with a final state constraint, in fact inequality (3) is considered in order to take into account sustainability issues, so that we will apply the appropriate transversality conditions.

The current value Hamiltonian functions related to each period are

$$H_i^c(x,u,\lambda_i) = \bar{P}_i u + P_{si} - C_e(x,u)u + \lambda_i(R-u),\ i \in \{1,2,3\}. \tag{7}$$

The necessary conditions for optimality are the following

- The restriction of the control function to each subinterval I_i maximizes the related Hamiltonian function, i.e.

$$u|_{I_i} = \arg\max_{u\in[0,\overline{u}]} H_i^c; \tag{8}$$

- The co-state functions $\lambda_i(t)$ satisfy

$$\dot{\lambda}_i(t) = -\frac{\partial H_i^c}{\partial x} + r\lambda_i(t) = \frac{\partial C_e(x,u)}{\partial x}u(t) + r\lambda_i(t); \tag{9}$$

the matching conditions

$$\lambda_2(T_1) = \lambda_1(T_1), \tag{10}$$

$$\lambda_3(T_2) = \lambda_2(T_2); \tag{11}$$

the transversality conditions

$$\lambda_3(T_3) \geq 0, \quad \lambda_3(T_3)(x(T_3) - x_0) = 0. \tag{12}$$

As

$$\frac{\partial H_i^c(x,u,\lambda_i)}{\partial u} = \bar{P}_i - \lambda_i(t) - \frac{\partial C_e(x,u)}{\partial u}u - C_e(x(t),u(t)) \tag{13}$$

and according to (4) and (5), we obtain the three inequalities

$$\frac{\partial^2 H_i^C}{\partial u^2} = -\frac{\partial^2 C_e(x,u)}{\partial u^2}u - 2\frac{\partial C_e(x,u)}{\partial u} < 0, \ i \in \{1,2,3\} \tag{14}$$

which guarantee the strict concavity of the current value Hamiltonian functions w.r.t. u. Consequently the three stationary points of H_i^C are the candidate optimal controls.

Nevertheless, the concavity of the Hamiltonian function w.r.t. (x,u) is not assured, therefore we cannot apply the Mangasarian sufficiency theorem [21] in this general setting and no other sufficiency result can be proved at this general stage.

The following Lemma gives some information about the state and the co-state functions.

Lemma 3.1 *The state function is monotonically strictly increasing and the co-state function associated to the third period vanishes at the final instant T_3, that is*

$$\lambda_3(T_3) = 0. \tag{15}$$

Proof. From the initial assumptions on the recharge rate and the bounding for the extraction rate ($u(t) \in [0,\overline{u}]$ and $R > \overline{u}$), we have that $\dot{x}(t) = R - u(t) > 0$, for all $t \in [0,T_3]$. This means that the water volume in the well $x(t)$ is monotonically strictly increasing and therefore at the final time it holds $x(T_3) > x_0$. From the transversality conditions (12) it trivially follows that $\lambda_3(T_3) = 0$. ■

4 Environmental costs depending on extraction rates: an example.

Let us now consider the particular unit cost function

$$C_e(x,u) = \frac{u}{x} \tag{16}$$

which satisfies the initial assumptions (4) and (5) and the limit condition $\lim_{x\to 0} C_e(x,u) = +\infty$, as stated in [23].

It is easy to prove that in this case the Hamiltonian functions $H_i^c(x,u,\lambda_i),\ i \in \{1,2,3\}$, are concave in (x,u), for each t $\in [0,T_3]$, so that the Mangasarian sufficiency Theorem holds and the necessary optimality conditions are also sufficient [21]. We have

$$\frac{\partial H^c(x(t),u(t),\lambda_i(t))}{\partial u(t)} = 0 \Leftrightarrow \bar{P} - \frac{\partial C_e(x(t),u(t))}{\partial u(t)} u(t) - C_e(x(t),u(t)) - \lambda_i(t) = 0,$$

from which it follows that

$$\bar{P}_i - \lambda_i(t) = \frac{u(t)}{x(t)} + \frac{u(t)}{x(t)}$$

and therefore the optimal extraction rate is

$$u^*(t) = \frac{\bar{P}_i - \lambda_i(t)}{2} x(t), \quad t \in I_i. \tag{17}$$

Observe that the optimal control is given in feedback form, as it depends on the value of the state function at each time. This feature is due to the non linearity of the Hamiltonian functions w.r.t. the control and represents a peculiarity of this model.

The co-state equations (9) become

$$\dot{\lambda}_i(t) = r\lambda_i(t) - \frac{u^2(t)}{x^2(t)} \Rightarrow \dot{\lambda}(t)_i = r\lambda_i(t) - \frac{(\bar{P}_i - \lambda_i(t))^2}{4} \tag{18}$$

and are to be solved backward, beginning from the last one, subject to the boundary condition (15), and imposing, in turn, the related matching condition as a boundary condition for the Cauchy problem of each further co-state function.

In what follows, we present the results of numerical simulations performed in order to solve the optimal control problem using the Generator Algebraic Models Simulator (GAMS). We use some technical and economic data taken from the Italian Veneto region and we estimate their value according to [5], to [4], and to projections of water prices and demand estimated by industry experts and local regulators (ATOs) on the basis of the Presidential Decree 1/8/1996 over the entire concession period. This estimations are consistent with the average revenues estimated for the provision of drinking water in Italy by CONVIRI [3], i.e. the national Authority in charge of controlling and supervising local regulators. Average revenues per cubic meter coincide with the tariff paid per cubic meter, therefore average revenues per cubic meter related to the use of surface water are equal to the groundwater ones.

Different scenarios may occur according to the parameters values. It turns out that u^* is piecewise continuous and depends on the initial groundwater stock. The higher the initial groundwater level, the more it is convenient to extract until the maximum extraction rate is reached (Figure 1, Figure 2).

Figure 1 gives an intuition of the optimal solution and illustrates the optimal groundwater extraction rates for different initial stock levels. The values used for simulations are the following: $r = 0.005$; $R = 8 \times 10^6\ m^3/month$; $\overline{u} = 6.2 \times 10^6\ m^3/month$; $\overline{P}_1 = 0.04\ Euro/m^3$; $\overline{P}_2 = 0.045\ Euro/m^3$; $\overline{P}_3 = 0.05\ Euro/m^3$; $P_{s1} = 0.13\ Euro/m^3$; $P_{s2} = 0.135\ Euro/m^3$; $P_{s3} = 0.16\ Euro/m^3$.

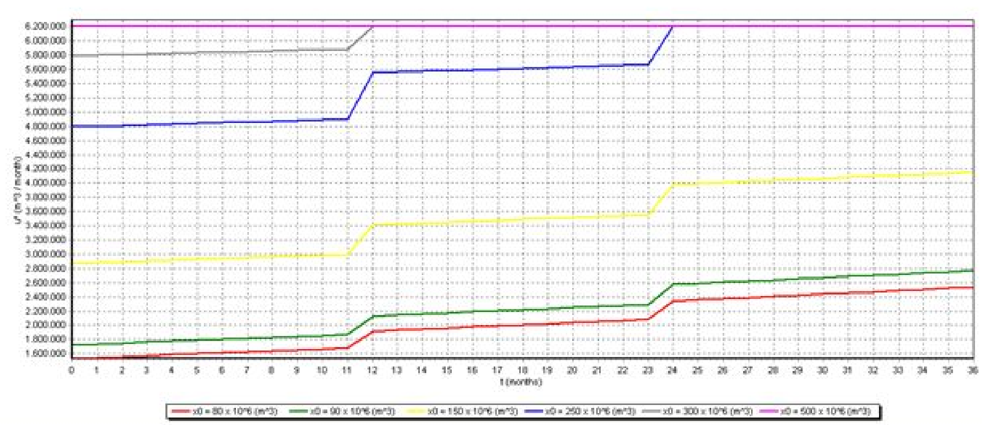

Figure 1: Optimal groundwater extraction rates for different x_0 ($\overline{P}_2 = 0.045\ Euro/m^3$)

We can observe that the greater the initial stock level, the sooner the maximum extraction rate is potentially reached. Moreover the greater the initial stock level, the greater the increment of the optimal extraction rate from one subinterval to the next. The extreme behaviour is obtained when the initial stock is very high (e.g. $x_0 = 800 \times 10^6\ m^3$) and the optimal extraction policy consists in extracting at the maximum rate (i.e. $6.2 \times 10^6\ m^3$/month) in the entire programming interval. In this case the difference between groundwater unit profits and surface water ones is monotonically increasing, i.e. $\overline{P}_1 < \overline{P}_2 < \overline{P}_3$, and it turns out that the optimal extraction rates are piecewise continuous and strictly monotonically increasing. The latter property might not hold whenever the profits difference is not monotonically increasing. For example, ceteris paribus, if $\overline{P}_2 = 0.06\ Euro/m^3$, then the optimal extraction rate turns out to be piecewise monotonically increasing as shown in Figure 2.

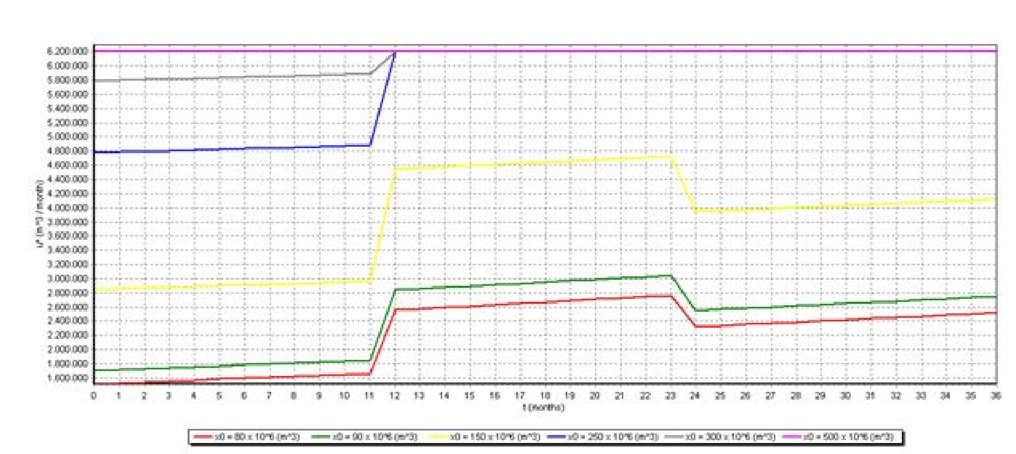

Figure 2: Optimal groundwater extraction rates for different x_0 ($\overline{P}_2 = 0.06\ Euro/m^3$)

It is worth noting that in this case, since $\overline{P}_2 > \overline{P}_1$ and $\overline{P}_2 > \overline{P}_3$, then it results significantly convenient to extract groundwater in the second period and therefore the extraction rates are greater than in the other two periods.

According to the simulations, it turns out that the optimal extraction rate depends on the recharge rate. If the gap between R and $\overline{u}$ is high, then the maximum extraction rate is reached sooner w.r.t. a lower gap, because in the former case the aquifer recharges faster.

In figure 3 it is represented a focus on the analysis w.r.t. the difference between groundwater and surface profits in the third period. We consider $x_0 = 120 \times 10^6\ m^3$ and assume that all the parameters remain the same as in Figure 1, apart from $\overline{P}_3$, which varies. For the sake of simplicity, the profits differences $\overline{P}_1$ and $\overline{P}_2$ are assumed as fixed and $\overline{P}_3 = 0.042;\ 0.045;\ 0.05;\ 0.06\ Euro/m^3$. Recalling that a high value of $\overline{P}_3$ signifies that the profit from surface water abstraction is very low (e.g. because of high operating costs due to sanitization and purification processes), it follows that the bigger the gap, the more it is optimal to extract from the groundwater source. Vice-versa the smaller the gap the more it is convenient surface water abstraction. In other words, when $\overline{P}_3 < \overline{P}_2$, surface water becomes relatively more convenient in the third period than in the second and the results are analogous to those illustrated in Figure 2 (i.e. the extraction rates are greater in the second period than in the third). Obviously when $\overline{P}_2 = \overline{P}_3$, the extraction rates are continuous and monotonically increasing in the interval $[T_2, T_3]$.

It is worth noting that the monotonicity of $u^*(t)$ is strongly affected by the sustainability assumption (1) that characterizes the Italian setting. In different contexts, where we can relax such an assumption, it turns out that the optimal control might be decreasing in some intervals.

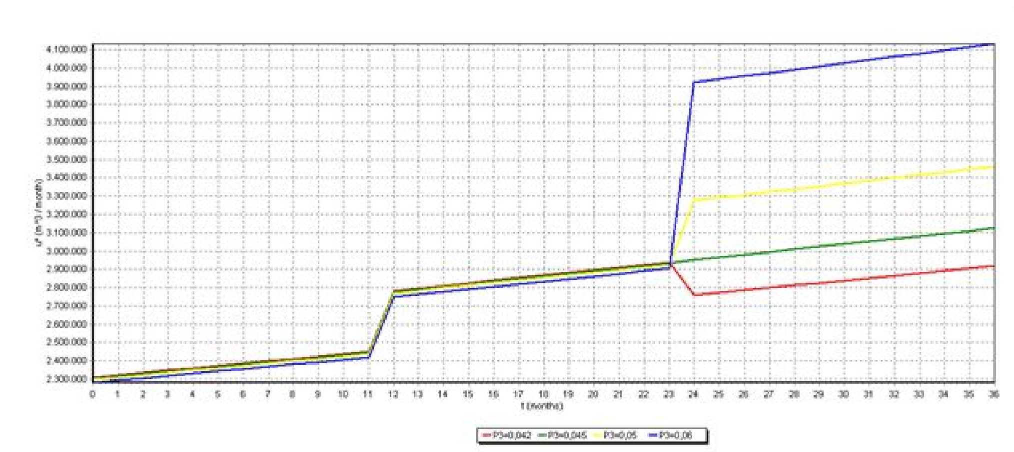

Figure 3: Optimal groundwater extraction rates for different $\overline{P}_3$

5 Environmental costs independent of extraction rates

As already mentioned, usually environmental unit costs are considered as not explicitly depending on the extraction rate. In this section we analyze this particular case in detail, studying possible scenarios that might occur, according to the initial water level. We expect that the linearity assumption on unit environmental costs will lead to pulsing solutions.

We assume that the environmental cost function $C_e(x)$ is differentiable and recalling the assumption of monotonicity w.r.t. the volume level x, we have

$$\frac{\partial C_e(x)}{\partial x} < 0.$$

Therefore the current value Hamiltonian function is

$$H_i^c(x,u,\lambda_i) = \bar{P}_i u + P_{si} - C_e(x)u + \lambda_i(R-u),\ \ i \in \{1,2,3\} \tag{19}$$

whose derivatives are

$$\frac{\partial H_i^c(x,u,\lambda_i)}{\partial u} = \bar{P}_i - C_e(x) - \lambda_i,$$

$$\frac{\partial H_i^c(x,u,\lambda_i)}{\partial x} = -\frac{\partial C_e(x)}{\partial x}u$$

and the first order optimality conditions become

$$\bar{P}_i - C_e(x(t) - \lambda_i(t) = 0, \quad i \in \{1,2,3\}.$$

The linearity of the Hamiltonian function w.r.t. the control leads to a bang-bang solution

$$u^*(t) = \begin{cases} 0, & if\ \bar{P} - \lambda_i(t) < C_e(x(t)) \Leftrightarrow (\bar{P} - C_e(x(t))) < \lambda_i(t), \\ \bar{u}, & if\ \bar{P} - \lambda_i(t) > C_e(x(t)) \Leftrightarrow (\bar{P} - C_e(x(t))) > \lambda_i(t). \end{cases}$$

As we assumed that the cost function $C_e(x)$ is monotonically strictly decreasing in x, so it is its inverse function and therefore we can rewrite the optimal control as follows

$$u^*(t) = \begin{cases} 0, & if\ x(t) < C_e^{-1}(\bar{P} - \lambda_i(t)), \\ \bar{u}, & if\ x(t) > C_e^{-1}(\bar{P} - \lambda_i(t)), \end{cases} \tag{20}$$

from which we can observe that low water levels require not to extract, while the optimal extraction rate is the allowed maximum one in case of high water levels. The optimal extraction rate is a pulsing one which switches according to the water level present in the aquifer at a given time. The motion equation has the following piecewise form

$$\dot{x}(t) = \begin{cases} R, & if\ u^*(t) = 0, \\ R - \bar{u}, & if\ u^*(t) = \overline{u}, \end{cases}$$

with boundary conditions

$$x(0) = x_0,$$

$$x(T_3) \geq x_0.$$

The co-state equations, together with the matching conditions (10), (11) and the transversality condition (15), constitute the following system

$$\begin{cases} \dot{\lambda}_i(t) = -\frac{\partial H^C}{\partial x} + r\lambda_i(t) = u(t) + r\lambda_i(t), \quad i \in \{1,2,3\}, \\ \lambda_2(T_1) = \lambda_1(T_1), \\ \lambda_3(T_2) = \lambda_2(T_2), \\ \lambda_3(T_3) = 0. \end{cases}$$

In other terms,

$$\dot{\lambda}_i(t) = \begin{cases} r\lambda_i(t), & if\ u^*(t) = 0, \\ r\lambda_i(t) + \frac{\partial C_e(x(t))}{\partial x(t)}\overline{u}, & if\ u^*(t) = \overline{u}. \end{cases} \tag{21}$$

The state and co-state equations are coupled together and constitute a two point boundary value problem that can be solved either forward (starting from the initial state condition) or backward (starting from the boundary co-state condition), once the cost function is known.

Let us now consider, as an explicative example, the following particular cost function

$$C_e(x) = \frac{1}{x} \tag{22}$$

independent from the extraction rate and which satisfies the condition $\lim_{x\to 0} C_e(x) = +\infty$ as stated in [23].
The optimal control rule is

$$u^*(t) = \begin{cases} 0, & if\ x(t) < \frac{1}{\bar{P}_i - \lambda_i(t)}, \\ \bar{u}, & if\ x(t) > \frac{1}{\bar{P}_i - \lambda_i(t)}. \end{cases} \tag{23}$$

Moreover, analyzing the sufficient optimality conditions, we can obtain the following results.

Lemma 5.1 *The co-state function $\lambda_3(t)$ is differentiable in $t = T_3$.*

Proof. Function $x(t)$ is monotonically strictly increasing as stated in Lemma 3.1. So we have

$$x(T_3) \geq x(t) \geq x_0 > 0,\ t \in [0, T]$$

and therefore

$$\lim_{t\to T_3} \dot{\lambda}_3(t) = \begin{cases} 0 & if\ u_3(T_3) = 0, \\ -\frac{\overline{u}}{(x_3(T_3))^2} < 0 & if\ u_3(T_3) = \overline{u}, \end{cases} \tag{24}$$

In both cases such limit is finite and therefore T_3 is a differentiability point for the co-state function $\lambda_3(t)$. ■

Let be $\overline{I}_3 = [T_3 - \epsilon, T_3] \subset I_3$ the last interval in which the co-state function is differentiable and the control function is constant. The following Lemma holds.

Lemma 5.2 *The co-state function $\lambda_3(t)$ is non-negative in the interval $\overline{I}_3$, i.e. $\lambda_3(t) \geq 0$ for all $t \in \overline{I}_3$.*

Proof. Let us assume that there exists a $\tilde{t} \in \overline{I}_3$ such that $\lambda_3(\tilde{t}) < 0$.

We can assume, without loss of generality, that $\tilde{t}$ is a differentiability point for λ_3, because the co-state function is piecewise differentiable, and therefore in any neighbourhood of $\tilde{t}$ there exist infinitely many differentiable points. Among such differentiability points there exists at least one $\tilde{t}$ such that $\lambda_3(t) < 0$.

Let us now consider separately the situations which occur at the final interval.

- If $u(t) = \overline{u}, \ \forall \ t \in \overline{I}_3$, the Cauchy problem associated to the co-state function is
$$\begin{cases} \dot{\lambda}_3(t) = r\lambda_3(t) - \frac{\overline{u}}{x^2(t)}, & \forall \ t \in \overline{I}_3, \\ \lambda_3(T_3) = 0, \end{cases} \tag{25}$$
from which we obtain $\dot{\lambda}_3(\tilde{t}) = r\lambda_3(\tilde{t}) - \frac{\overline{u}}{x_3^2(\tilde{t})} < 0$, and therefore $\lambda_3(t) < \lambda_3(\tilde{t}) \leq 0, \ \forall \ t \in [\tilde{t}, \tilde{t}+\epsilon]$, $\epsilon > 0$. Reiterating and exploiting the continuity hypothesis on the co-state function we have $\dot{\lambda}_3(t) = r\lambda_3(t) - \frac{\overline{u}}{x_3^2(t)} < 0$, for all $t \geq \tilde{t}$ and therefore also in $t = T_3$. This contradicts the transversality condition (15) according to which $\lambda(T_3) = 0$.

- If $u(t) = 0, \ \forall \ t \in \overline{I}_3$, then the Cauchy problem associated to the co-state function is
$$\begin{cases} \dot{\lambda}_3(t) = r\lambda_3(t), & \forall \ t \in \overline{I}_3, \\ \lambda_3(T_3) = 0, \end{cases} \tag{26}$$
which solved trivially gives $\lambda_3(t) = 0, \ \forall \ t \in \overline{I}_3$.

■

The results of the previous Lemma can be extended to the entire programming interval $[0, T_3]$.

Lemma 5.3 *The co-state functions are positive in the programming interval* $[0, T_3]$, *that is* $\lambda_i(t) \geq 0$ *for all* $t \in [0, T_3]$, $\ i \in \{1, 2, 3\}$.

Proof. Let us assume that there exists a differentiability point for λ_i, $\tilde{t} \in [0, T_3]$ such that $\lambda_i(\tilde{t}) < 0$.

Let us now consider separately the situations which occur at the point $\tilde{t}$.

- If $u(\tilde{t}) = 0$ then $\dot{\lambda}_i(\tilde{t}) = r\lambda_i(\tilde{t}) < 0$ and therefore $\lambda_i(t) < \lambda_i(\tilde{t}) = 0, \ \forall \ t \in [\tilde{t}, \tilde{t} + \epsilon]$.

- If $u(\tilde{t}) = \overline{u}$ then $\dot{\lambda}_i(\tilde{t}) = r\lambda_i(\tilde{t}) - \frac{\overline{u}}{x_3^2(\tilde{t})} < 0$, and therefore $\lambda_i(t) < \lambda_i(\tilde{t}) = 0, \ \forall \ t \in [\tilde{t}, \tilde{t} + \epsilon]$.

Reiterating and exploiting the continuity hypothesis on the co-state function we have $\dot{\lambda}_i(t) = r\lambda_i(t) - \frac{\overline{u}}{x_3^2(t)} < 0$, for all $t \geq \tilde{t}$ and therefore $\lambda_3(t) < 0$ for $t \in I_3$. This contradicts the previous Lemma. ∎

In case the optimal solution requires not to extract at the end of the planning period, then it is not convenient to extract at all, as the following proposition states.

Proposition 5.1 *If $u(t) = 0$, for $t \in \overline{I}_3$, then $u(t) = 0$, for all $t \in [0, T_3]$.*

Proof. If $u(t) = 0$, for $t \in \overline{I}_3$, then in the differentiability points of such interval it holds

$$\begin{cases} \dot{\lambda}_3(t) = r\lambda_3(t), \\ \lambda_3(T_3) = 0, \end{cases} \tag{27}$$

from which

$$\lambda_3(t) = 0, \ \forall t \in \overline{I}_3.$$

Furthermore we have that

$$\dot{\lambda}_3(T_3 - \epsilon) = r\lambda_3(T_3 - \epsilon) = 0$$

so, reiterating with continuity, we can prove that $\lambda(t) = 0$, for all $t \in [0, T_3]$.

The state function is monotonically increasing so that

$$x(t) < x(T_3) = \frac{1}{\bar{P}_3 - \lambda_3(T_3)} = \frac{1}{\bar{P}_3 - \lambda_3(t)} = \frac{1}{\bar{P}_3}, \quad t \in [T_2, T_3],$$

and therefore $u(t) = 0$, for $t \in [T_2, T_3]$, in particular $u(T_2) = 0$.

Analogously we can extended the same reasoning to the previous periods and prove that $u(t) = 0$, for all $t \in [0, T_3]$. ∎

According to proposition 5 it is never convenient to extract at the beginning of the planning period and then to stop extracting. In case it is convenient to wait and postpone the extraction.

Observe that according to the range of parameters values in the Italian setting, it always results the trivial solution $u^*(t) = \overline{u}$, for all $t \in [0, T_3]$. This fact in some sense makes more interesting the choice of a unit environmental cost function depending on the extraction rate, as discussed in Section 4.

6 Concluding Remarks

Although drinking water supply management is typically modelled as a single source serving a group of consumers, resource providers must often decide to manage multiple sources simultaneously. The interconnection of water supply sources allows the provider to increase the concession contract value (i.e. maximize its profits) by strategically decide the optimal conjunctive use of multiple supply sources. If, on the one hand, the interconnection of multiple sources involves high irreversible sunk costs, on the other hand it protects users against uncertainties in service provision by generating operational flexibility. This flexibility can be *de facto* economically relevant, in fact it contributes to profit maximization and hedging of risks and the joint use of two sources (one of which is stock and the other is flow) can therefore lead to a cheaper supply than that gained by their independent use.

In this paper we formulate and solve an optimal control model in order to determine the optimal abstraction policy for a water service provider who has invested in the interconnection of groundwater and surface water supply sources. The novelty of the paper resides in the modelling of the optimal control problem in a three-period finite horizon where environmental extraction costs matters and depend both on the groundwater stock and the extraction rate. We share with the existing literature the realistic assumption on the monotonicity of the environmental cost w.r.t. the water stock but we also assume that environmental costs are convex w.r.t. the extraction rate and the recharge rate is constant. In the case where environmental unit costs are equal to the ratio between the extraction rate and the stock level, the optimal control is given in a feedback form. These results are supported by numerical simulations performed using technical and economic data related to the Italian Veneto region. According to the simulations, it turns out that the optimal extraction rate depends on the recharge rate and the higher the difference between the recharge rate and the extraction rate, the sooner it is reached the maximum extraction rate. Moreover, in some circumstances, regardless environmental extraction costs, the greater the difference between unit profits from groundwater extraction and surface water abstraction, the more it is convenient to use groundwater to supply water to final users. We also analyse the case where environmental costs are independent of extraction rates and a bang-bang solution is obtained. Further developments will take into consideration variable recharge rates.

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