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Convergence Rates in Regularization for Nonlinear Ill-Posed Equations with Perturbative Data

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Abstract

The purpose of this paper is to study the variational variant of Tikhonov's regularization method for solving a system of nonlinear ill-posed equations in real Hilbert spaces under perturbative operators and right hand sides, then some numerical experiments are given to illustrate convergence rates of regularized solution for solving a system of nonlinear ill-posed equations in real Hilbert spaces under perturbative operators and right hand sides.

Keywords: Hilbert spaces, Tikhonov regularization, nonexpansive mappings

1. INTRODUCTION

Let X, Y_j be Hilbert spaces with the scalar product and norm of X denoted by the symbols $\langle .,. \rangle_X$ and $\|.\|_X$, respectively. Let $A_j, j = 1, ..., N$, be N nonlinear operators from a closed convex subset $\mathcal{D} \subseteq X$ into Y_j having the following properties:

(i) A_j are continuous;

(ii) A_j are weakly closed, i.e., for any sequence $\{x_n\} \subset \mathcal{D}$ converging weakly in X to x such that $A_j(x_n)$ converges weakly in Y_j to y, it implies that $A_j(x) = y$.

Consider the following problem: find an element $x_0 \in \mathcal{D}$ such that

$$A_j(x_0) = f_j, \quad \forall j = 1, ..., N, \tag{1.1}$$

where f_j are given in Y_j a priori. Set

$$S_j = \{\overline{x} \in X : A_j(\overline{x}) = f_j\}, j = 1, ..., N, \quad S = \cap_{j=1}^N S_j.$$

Here, we suppose that $S \neq \emptyset$. From the properties of A_j it is easy to see that S_j are closed in X. Therefore, S is also closed.

The particular case of (1.1) where $X = Y_j = H$, a Hilbert space, $A_j = I - T_j + f_j$ with $\|T_j(x) - T_j(y)\|_H \leq \|x - y\|_H \forall x, y \in \mathcal{D}$ and I is the identity operator in H, was studied in [1]-[3] and applied to solve the image recovery problems in [4]-[6].

The system of equations (1.1) can be written in the form

$$\mathcal{A}(x) = f, \tag{1.2}$$

where $\mathcal{A} : X \to Y := Y_1 \times ... \times Y_N$ by $\mathcal{A}(x) := (A_1(x), ..., A_N(x))$, and $f := (f_1, ..., f_N)$. Note that (1.2) can be seen as a special case of (1.1) with $N = 1$. Howerver, one potential advantage of (1.1) over (1.2) can be that it might better reflect the structure of the underlying information $(f_1, ..., f_N)$ leading to the couplet system, than a plain concantenation into one single data element f could. The second advantage is that in estimating convergence rates of regularization solution which is showed later we need only the smooth condition for one among A_j, while for (1.2) we need the condition for all $A_j, j = 1, ..., N$.

With the above properties on A_j, each j-equation (1.1) is ill-posed. By this we mean that the solution set S_j does not depend continuously on the data (A_j, f_j). Therefore, to find a solution of each j-equation in (1.1) we have to use stable methods. One of those methods is the variational variant of Tikhonov's regularization that consists of minimizing the functional

$$\|A_j^h(x) - f_j^\delta\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2 \tag{1.3}$$

where x_* is some element in $X \setminus S_j$, $\alpha > 0$ is the small parameter of regularization, f_j^δ are the approximations for f_j such that $\|f_j - f_j^\delta\|_{Y_j} \leq \delta, \delta \to 0$, and A_j^h are approximations for A_j in the sense

$$\|A_j(x) - A_j^h(x)\|_{Y_j} \leq hg(\|x\|_X), \tag{1.4}$$

with the bounded (image of bounded set is bounded) nonegative function $g(t), t \geq 0$. Assume that A_j^h have the same properties as A_j. For the case where $A_j^h \equiv A_j$, it proved in [7] that each j-minimization problem in (1.3) has unique solution $x_j^{\alpha\delta}$, and if $\delta^2/\alpha, \alpha \to 0$ then $\{x_j^{\alpha\delta}\}$ contains a converging subsequence, and the limit point x_j of any converging subsequence has the following property

$$\|x_j - x_*\|_X = \min_{x \in S_j} \|x - x_*\|_X, \quad j = 1, ..., N.$$

Our problem: find $x_\alpha^{h\delta}$ such that $x_\alpha^{h\delta} \to x_0$ as $h, \delta, \alpha \to 0$, a relation $\alpha = \alpha(h, \delta)$ such that $x_{\alpha(h,\delta)}^{h\delta} \to x_0$ as $h, \delta \to 0$, and finally estimate the value $\|x_{\alpha(h,\delta)}^{h,\delta} - x_0\|$ where x_0 is a x_*-minimal norm element in S (x_*-MNS).

In [8], we considered the case that $A_j, j = 1, ..., N$ are given exactly. In this paper, we consider the general case that A_j and f_j are given approximately by A_j^h and f_j^δ, respectively. For this purpose, we consider the problem

$$\sum_{j=1}^{N} \|A_j^h(x) - f_j^\delta\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2 \to \min_{\mathcal{D}}. \tag{1.5}$$

Notice that when $Y_j = X^*$, the conjugate of the Banach spaces X, and A_j are the derivatives of some weakly lower semicontinuous and proper convex functionals, the stated problem was considered in [9] based on the theory of monotone mappings.

Below, the symbols $\rightharpoonup$ and $\to$ denote the weak convergence and convergence in the norm, respectively, and $a \sim b$ is meant $a = O(b)$ and $b = O(a)$.

2. MAIN RESULTS

Under the assumptions on A_j^h it can be easy to show that problem (1.5) admits a solution. Since A_j^h are nonlinear, the solution will not be unique, in general.

Theorem 2.1. *Let $\alpha > 0$, $h_k \to h > 0$ ($A_j^{h_k}$ approximate A_j^h in sense* (1.4)), *$f_j^{\delta_k} \to f_j^\delta$ with $\delta \geq 0$, and x_k be a minimizer of* (1.5) *with f_j^δ and A_j^h replaced by $f_j^{\delta_k}$ and $A_j^{h_k}$, respectively. Then there exists a convergent subsequence of $\{x_k\}$ and the limit of every convergent subsequence is a minimizer of* (1.5).

Proof. For any $x \in \mathcal{D}$ we have

$$\sum_{j=1}^{N} \|A_j^{h_k}(x_k) - f_j^{\delta_k}\|_{Y_j}^2 + \alpha\|x_k - x_*\|_X^2 \leq \sum_{j=1}^{N} \|A_j^{h_k}(x) - f_j^{\delta_k}\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2. \tag{2.1}$$

Hence, $\|x_k\|_X$ and $\|A_j^{h_k}(x_k)\|_{Y_j}$ are bounded for each j. Consequence, there exist a subsequence $\{x_m\}$ of $\{x_k\}$ and $\tilde{x}$ such that

$$x_m \rightharpoonup \tilde{x}, A_j^{h_m}(x_m) \rightharpoonup A_j^h(\tilde{x}), j = 1, ..., N.$$

By the weak lower semicontinuity of the norm and the weakly closed property of A_j^h we have

$$\|\tilde{x} - x_*\|_X \leq \liminf \|x_m - x_*\|_X$$

and

$$\|A_j^h(\tilde{x}) - f_j^\delta\|_{Y_j} \leq \liminf \|A_j^{h_m}(x_m) - f_j^{\delta_m}\|_{Y_j}.$$

Therefore,

$$\sum_{j=1}^N \|A_j^h(\tilde{x}) - f_j^\delta\|_{Y_j}^2 \leq \sum_{j=1}^N \liminf \|A_j^{h_m}(x_m) - f_j^{\delta_m}\|_{Y_j}^2. \tag{2.2}$$

Moreover, (2.1) implies

$$\begin{aligned}\sum_{j=1}^N \|A_j^h(\tilde{x}) - f_j^\delta\|_{Y_j}^2 + \alpha\|\tilde{x} - x_*\|_X^2 \leq& \sum_{j=1}^N \liminf \|A_j^{h_m}(x_m) - f_j^{\delta_m}\|_{Y_j}^2 \\ &+ \alpha \liminf \|x_m - x_*\|_X^2 \\ &\leq \sum_{j=1}^N \limsup \|A_j^{h_m}(x_m) - f_j^{\delta_m}\|_{Y_j}^2 + \alpha \limsup \|x_m - x_*\|_X^2 \\ &\leq \sum_{j=1}^N \lim \|A_j^{h_m}(x) - f_j^{\delta_m}\|_{Y_j}^2 + \alpha \lim \|x - x_*\|_X^2 \\ &= \sum_{j=1}^N \|A_j^h(x) - f_j^\delta\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2\end{aligned}$$

for all $x \in \mathcal{D}$. This implies that $\tilde{x}$ is a minimizer of (1.5) and that

$$\begin{aligned}\lim_{m\to\infty}\Big(\sum_{j=1}^N \|A_j^{h_m}(x_m) - f_j^{\delta_m}\|_{Y_j}^2 + \alpha\|x_m - x_*\|_X^2\Big) \\ = \sum_{j=1}^N \|A_j^h(\tilde{x}) - f_j^\delta\|_{Y_j}^2 + \alpha\|\tilde{x} - x_*\|_X^2.\end{aligned} \tag{2.3}$$

Now, assume that $x_m \not\to \tilde{x}$. Then $c := \limsup \|x_m - x_*\|_X > \|\tilde{x} - x_*\|_X$ and there exists a subsequence $\{x_n\}$ of $\{x_m\}$ such that $x_n \rightharpoonup \tilde{x}, A_j^{h_n}(x_n) \rightharpoonup A_j^h(\tilde{x})$

and $\|x_n - x_*\|_X \to c$. As a consequence of (2.3), we obtain

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=1}^{N}\|A_j^{h_n}(x_n) - f_j^{\delta_n}\|_{Y_j}^2 \\ &= \sum_{j=1}^{N}\|A_j^h(\tilde{x}) - f_j^{\delta}\|_{Y_j}^2 + \alpha(\|\tilde{x} - x_*\|_X^2 - c^2) \\ &< \sum_{j=1}^{N}\|A_j^h(\tilde{x}) - f_j^{\delta}\|_{Y_j}^2\end{aligned}$$

in contradiction to (2.2). This argument shows that $x_m \to \tilde{x}$.

Theorem 2.2. *Let $\alpha(h,\delta)$ be such that $\alpha(h,\delta) \to 0, \delta^2/\alpha(h,\delta) \to 0$ and $h^2/\alpha(h,\delta) \to 0$ as $h \to 0, \delta \to 0$. Then every sequence $\{x_{\alpha_k}^{h_k\delta_k}\}$, where $h_k \to 0, \delta_k \to 0, \alpha_k = \alpha(h_k,\delta_k)$ and $x_{\alpha_k}^{h_k\delta_k}$ is a solution of (1.5), has a convergent subsequence. The limit of every convergent subsequence is an x_*-MNS. If, in addition, the x_*-MNS x_0 is unique, then*

$$\lim_{h,\delta\to 0} x_{\alpha(h,\delta)}^{h\delta} = x_0.$$

Proof. From (1.5) we have

$$\begin{aligned}&\sum_{j=1}^{N}\|A_j^h(x_{\alpha(h,\delta)}^{h\delta}) - f_j^{\delta}\|_{Y_j}^2 + \alpha(h,\delta)\|x_{\alpha(h,\delta)}^{h\delta} - x_*\|_X^2 \\ &\quad\le \sum_{j=1}^{N}\|A_j^h(x) - f_j^{\delta}\|_{Y_j}^2 + \alpha(h,\delta)\|x - x_*\|_X^2 \\ &\quad\le \sum_{j=1}^{N}(\|A_j^h(x) - A_j(x)\|_{Y_j} + \|f_j - f_j^{\delta}\|_{Y_j})^2 + \alpha(h,\delta)\|x - x_*\|_X^2 \\ &\quad\le N(hg(\|x\|_X) + \delta)^2 + \alpha(h,\delta)\|x - x_*\|_X^2, \quad x \in S.\end{aligned} \tag{2.4}$$

Hence,

$$\|x_{\alpha(h,\delta)}^{h\delta} - x_*\|_X^2 \le N\frac{(hg(\|x\|_X) + \delta)^2}{\alpha(h,\delta)} + \|x - x_*\|_X^2 \quad \forall x \in S. \tag{2.5}$$

Consequently, $\{x_{\alpha(h,\delta)}^{h\delta}\}$ is bounded, i.e. there exists a positive constant d_1 such that $\|x_{\alpha(h,\delta)}^{h\delta}\| \le d_1$. Therefore, every sequence $\{x_{\alpha_k}^{h_k\delta_k}\}$, where $h_k \to 0, \delta_k \to 0, \alpha_k = \alpha(h_k,\delta_k)$ and $x_{\alpha_k}^{h_k\delta_k}$ is a solution of (1.5), has a weak convergent subsequence. Let $\{x_{\alpha_m}^{h_m\delta_m}\} \subset \{x_{\alpha_k}^{h_k\delta_k}\}$ be such that $x_{\alpha_m}^{h_m\delta_m} \rightharpoonup \overline{x}$ as $m \to \infty$. From (2.4) we see that

$$\begin{aligned}\|A_j^{h_m}(x_{\alpha_m}^{h_m\delta_m}) - f_j^{\delta_m}\|_{Y_j}^2 &\le N(h_m g(\|x\|_X) + \delta_m)^2 \\ &+\alpha_m\|x - x_*\|_X^2, 1 \le j \le N.\end{aligned} \tag{2.6}$$

First, note that $A_j^{h_m}(x_{\alpha_m}^{h_m\delta_m}) \rightharpoonup A_j(\tilde{x})$ as $m \to \infty$. Indeed, this follows from

$$|\langle A_j(x_m) - A_j^{h_m}(x_m), y^*\rangle| \le Ch_m \forall y^* \in Y_j^*,$$

where $C = \sup\{g(t) : 0 \le t \le d_1\}, d_1 \ge \|x_m\|$, and $A_j(x_m) \rightharpoonup A_j(\overline{x})$. Tending $m \to \infty$ in (2.6), we obtain $\|A_j(\overline{x}) - f_j\|_{Y_j} = 0$, i.e. $\overline{x} \in S_j, 1 \le j \le N$.

From (2.5) it implies that $\overline{x}$ is a x_*-MNS of (1.1), and $\|x_{\alpha_m}^{h_m\delta_m} - x_*\|_X \to \|\overline{x} - x_*\|_X$. Since X is a Hilbert space, then $x_{\alpha_m}^{h_m\delta_m} \to \overline{x}$ as $m \to \infty$. Theorem is proved.

Now, assume that $x_{\alpha(h,\delta}^{h\delta} \to x_0$, as $h, \delta \to 0$. The convergence rates of $\{x_{\alpha(h,\delta)}^{h\delta}\}$ is defined by the following theorem.

Theorem 2.3. *Let the following conditions hold:*

(i) A_1 is Fréchet differentiable

(ii) there exists $L > 0$ such that $\|A_1'(x_0) - A_1'(z)\|_{Y_j} \le L\|x_0 - z\|_X$ for z in some neighbouhood $\mathcal{U}$ of x_0

(iii) there exists $\omega \in Y_1$ such that $x_0 - x_* = A_1'(x_0)^*\omega$

(iv) $L\|\omega\| < 1$.

Then for the choice $\alpha \sim (h+\delta)^p, 0 < p < 2$, we obtain

$$\|x_{\alpha(h,\delta)}^{h\delta} - x_0\|_X = O((h+\delta)^{1-\frac{p}{2}}).$$

Proof. Using (2.4) with $x = x_0$ we obtain

$$\begin{aligned}\sum_{j=1}^{N}\|A_j^h(x_{\alpha(h,\delta)}^{h\delta}) - f_j^\delta\|_{Y_j}^2 + \alpha(h,\delta)\|x_{\alpha(h,\delta)}^{h\delta} - x_0\|_X^2 \\ \le N(hg(\|x_0\|_X) + \delta)^2 \\ + \alpha(h,\delta)(\|x_0 - x_*\|_X^2 - \|x_{\alpha(h,\delta)}^{h\delta} - x_*\|_X^2 + \|x_{\alpha(h,\delta)}^{h\delta} - x_0\|_X^2).\end{aligned}$$

Hence,

$$\begin{aligned}\|A_1(x_{\alpha(h,\delta)}^{h\delta}) - f_1^\delta\|_{Y_j}^2 + \alpha(h,\delta)\|x_{\alpha(h,\delta)}^{h\delta} - x_0\|_X^2 \le N(hg(\|x_0\|_X) + \delta)^2 \\ + 2\alpha(h,\delta)\langle\omega, A_1'(x_0)(x_0 - x_{\alpha(h,\delta)}^{h\delta})\rangle.\end{aligned} \tag{2.7}$$

Note that condition (i) and (ii) imply

$$A_1(x_{\alpha(h,\delta)}^{h\delta}) = A_1(x_0) + A_1'(x_0)(x_{\alpha(h,\delta)}^{h\delta} - x_0) + r_\alpha^{h\delta} \tag{2.8}$$

with

$$\|r_\alpha^{h\delta}\| \le \frac{1}{2}L\|x_{\alpha(h,\delta)}^{h\delta} - x_0\|^2. \tag{2.9}$$

Combining (2.7)-(2.9) leads to

$$\begin{aligned}
&\|A_1^h(x^{h\delta}_{\alpha(h,\delta)}) - f_1^\delta\|^2_{Y_j} + \alpha(h,\delta)\|x^{h\delta}_{\alpha(h,\delta)} - x_0\|^2_X \le N(hg(\|x_0\|_X) + \delta)^2 \\
&\quad - 2\alpha(h,\delta)\langle \omega, (f_1 - f_1^\delta) + (f_1^\delta - A_1(x^{h\delta}_{\alpha(h,\delta)}) + r^{h\delta}_\alpha\rangle \\
&\quad \le N(hg(\|x_0\|_X) + \delta)^2 \\
&\quad + 2\|\omega\|\alpha(h,\delta)\delta + 2\|\omega\|\alpha(h,\delta)(\|A_1^h(x^{h\delta}_{\alpha(h,\delta)}) - f_1^\delta\|_{Y_j} + C_0 h) \\
&\quad + \alpha(h,\delta)L\|\omega\|x^{h\delta}_{\alpha(h,\delta)}) - x_0\|^2_X
\end{aligned}$$

and hence

$$\begin{aligned}
&\|A_1^h(x^{h\delta}_{\alpha(h,\delta)}) - f_1^\delta\|^2_{Y_j} + \alpha(h,\delta)(1 - L\|\omega\|)\|x^{h\delta}_{\alpha(h,\delta)} - x_0\|^2_X \le \\
&N(hg(\|x_0\|_X) + \delta)^2 + 2\|\omega\|\alpha(h,\delta)(h+\delta) \\
&+ 2\|\omega\|\alpha(h,\delta)\|A_1^h(x^{h\delta}_{\alpha(h,\delta)}) - f_1^\delta\|_{Y_j}.
\end{aligned} \tag{2.10}$$

Together with (iv) and the implication

$$(a, b, c \ge 0, a^2 \le ab + c^2) \Rightarrow a \le b + c$$

(2.10) implies

$$\begin{aligned}
\|A_1^h(x^{h\delta}_{\alpha(h,\delta)}) - f_1^\delta\|_{Y_j} \le &\left[N(hg(\|x_0\|_X) + \delta)^2 + 2\|\omega\|\alpha(h,\delta)(h+\delta)\right]^{1/2} \\
&+ 2\|\omega\|\alpha(h,\delta).
\end{aligned}$$

Together with (2.9), this implies

$$\|x^{h\delta}_{\alpha(h,\delta)} - x_0\|_X \le \frac{(N(hg(\|x_0\|_X) + \delta)^2 + 2\|\omega\|\alpha(h,\delta)(h+\delta))^{1/2}}{\sqrt{\alpha(h,\delta)}(1 - L\|\omega\|)^{1/2}}. \tag{2.11}$$

The assertion now follows from (2.11), if $\alpha(h,\delta) \sim (h+\delta)^p, 0 < p < 2$.

3. NUMERICAL EXAMPLES

For illustration, we consider the following problem of finding a common solution of two systems of nonlinear equations

$$A_j(x) = f_j, \quad \forall j = 1, 2, \tag{3.1}$$

where

$$A_1(x) = f_1 \Leftrightarrow \begin{cases} x_1^4 + x_2^4 = 2 \\ x_1^2 - x_2^2 = 0 \\ x_1 x_3 = 0 \end{cases}$$

$$A_2(x) = f_2 \Leftrightarrow \begin{cases} x_1 - \ \ x_2 + x_3 \ = 0 \\ 2x_1 - 2x_2 + x_3 = 0 \\ x_1 - \ \ x_2 - x_3 = 0 \end{cases}$$

It is easy to verify that $A_1(x) = f_1$ posses four solutions

$$s_1 = (-1; -1; 0),\ s_2 = (-1; 1; 0),\ s_3 = (1; -1; 0),\ s_4 = (1; 1; 0),$$

A_2 is a linear operator, $rank(A_2) = 2$, so system (3.1) posses two solutions

$$s_1 = (-1; -1; 0),\ \ s_4 = (1; 1; 0).$$

We consider the case that A_j and f_j are given approximatly by A_j^h and f_j^δ, respectivly. Where

$$f_j^\delta = f_j + \Delta,\ \ \Delta = (\delta/\sqrt{3}; \delta/\sqrt{3}; \delta/\sqrt{3}), j = 1, 2.$$

$$A_1^h(x) = \begin{cases} (1+h)x_1^4 + x_2^4 \\ x_1^2 - (1-h)x_2^2 \\ (1+h)x_1x_3 \end{cases}$$

$$A_2^h(x) = \begin{cases} (1+h)x_1 - x_2 + x_3 \\ 2x_1 - (2+h)x_2 + (1+h)x_3 \\ x_1 - x_2 - (1+h)x_3 \end{cases}$$

Based on results in section 2, the common solution of (3.1) can be found by solving the following optimization problem

$$F(x) = \left\|A_1^h(x) - f_1^\delta\right\|_{R^3}^2 + \left\|A_2^h(x) - f_2^\delta\right\|_{R^3}^2 + \alpha \left\|x - x_*\right\|_{R^3}^2 \to \min, \quad (3.2)$$

where $\|x\|_{R^3} = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for every $x = (x_1; x_2; x_3) \in R^3$. It is not difficult to verify that (3.2) is equivalent to the following equation

$$g_i = \frac{\partial F}{\partial x_i} = 0,\ i = 1, 2, 3. \quad (3.3)$$

The following table 1 and table 2 show the calculation results for the approximation solution. Regularized equation (3.3) can be solved by using Newton's iterative method [10]

$$x^{(k+1)} = x^{(k)} - J(x^{(k)})^{-1}G(x^{(k)}), x^{(0)} = x_* = (x_1^*; x_2^*; x_3^*),$$

where

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{pmatrix},$$

Table 1 shows approximation solutions with iteration number is 1000,

$$x^{(0)} = x_* = (5, 5, 5), \alpha = h + \delta.$$

$h = \delta$	α	x_1	x_2	x_3	$\left\lVert x^{h\delta}_{\alpha(h,\delta)} - s_1 \right\rVert_{R^3}$	$\left\lVert x^{h\delta}_{\alpha(h,\delta)} - s_4 \right\rVert_{R^3}$
1.0000	2.0000	0.9574	1.1537	0.9288	3.0549	0.9424
0.1000	0.2000	0.9993	1.0317	0.2491	2.8614	0.2511
0.0100	0.0200	1.0002	1.0036	0.0291	2.8313	0.0293
0.0010	0.0020	1.0000	1.0004	0.0030	2.8287	0.0030
0.0001	0.0002	1.0000	1.0000	0.0003	2.8285	0.0003
0.0000	0.0000	1.0000	1.0000	0.0000	2.8284	0.0000

Table 2 shows approximation solutions with iteration number is 1000,

$$x^{(0)} = x_* = (-5, -5, -5), \alpha = h + \delta.$$

$h = \delta$	α	x_1	x_2	x_3	$\left\lVert x^{h\delta}_{\alpha(h,\delta)} - s_1 \right\rVert_{R^3}$	$\left\lVert x^{h\delta}_{\alpha(h,\delta)} - s_4 \right\rVert_{R^3}$
1.0000	2.0000	-0.7916	-1.2890	-1.0080	1.0692	3.0766
0.1000	0.2000	-0.9780	-1.0519	-0.2465	0.2528	2.8607
0.0100	0.0200	-0.9980	-1.0059	-0.0285	0.0291	2.8313
0.0010	0.0020	-0.9998	-1.0006	-0.0029	0.0030	2.8287
0.0001	0.0002	-1.0000	-1.0001	-0.0003	0.0003	2.8285
0.0000	0.0000	-1.0000	-1.0000	-0.0000	0.0000	2.8284

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