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Radially Symmetric Internal Layers for an Inhibitory System

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Abstract

We are concerned with an activator-inhibitor system on an n-dimensional ball such that the inhibitor is activated by an activator as well as the spatial average of its inhibitor. We analyze the existence of the radially symmetric solutions and the occurrence of Hopf bifurcation in the interfacial problem as the bifurcation parameters vary.

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1 Introduction

This paper is concerned with the following activator-inhibitor system such that the inhibitor is activated by not only an activator u but also its own spatial average $< v >$ ([9]):

$$\begin{cases} \sigma\varepsilon\, u_t - \varepsilon^2\,\nabla^2 u = f(u,v) \equiv -u + H(u-a_0) - v, \\ v_t - \nabla^2 v = g(u,v) \equiv u - \mu v + (\mu - 1) < v >, \;\; t > 0, \mathbf{x} \in \Omega, \\ 0 = \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}}, \;\; \mathbf{x} \in \partial\Omega, \end{cases} \tag{1}$$

where ε, σ, μ and a_0 are positive constant parameters and H is a Heaviside step function. The domain $\Omega = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < R\}$ is the ball in n-dimensional space, and $\mathbf{n}$ stands for the unit outward normal on the boundary $\partial\Omega$. The spatial average $< v >$ is defined by $< v >= \frac{1}{|\Omega|} \int_\Omega v\, d\Omega$, where $|\Omega|$ is the measure of Ω.

The purpose of this paper is to explore the dynamics of interfaces in the problem (1) and we shall compare it with the result of [3]. Suppose that there is only one $(n-1)$

-dimensional hypersurface $\eta(t)$ which is simply single closed curve given in the domain Ω in such a way that $\Omega = \Omega_+(t) \cup \eta(t) \cup \Omega_-(t)$, where $\Omega_-(t) = \{\mathbf{x} \in (0, R) \ : u(\mathbf{x}, t) > a_0\}$ and $\Omega_+(t) = \{\mathbf{x} \in (0, R) \ : u(\mathbf{x}, t) < a_0\}$. The equation of $\eta(t)$ is given by (see [6, 8]):

$$\frac{d\,\eta(t)}{dt} \cdot \nu = C(v_i), \ \mathbf{x} \in \eta(t),$$

where ν is the outward normal vector on $\eta(t)$, v_i is the value of v on the interface $\eta(t)$, and $C(v)$ is the velocity of the interface. Based on the result of [1, 5, 7, 10], the trajectory with a unique value of $C = C(v_0)$ exists which is given by $C(v_0) = -(h^+(v_0) - 2h^0(v_0) + h^-(v_0))$ where h^+, h^0 and h^-, are the triple valued functions of u in the equation $f(u, v) = 0$. Furthermore, the velocity of the interface $C(v)$ is a continuously differentiable function defined on an interval $\mathbf{I} := (-a_0, 1 - a_0)$ and thus it can be normalized by

$$C(v(\eta)) = -\frac{1}{\sigma} \frac{1 - 2a_0 - 2v(\eta)}{\sqrt{(v(\eta) + a_0)(1 - a_0 - v(\eta))}}.$$

An analysis of the dynamics of this process has been shown (see example [1, 7]) to lead a free boundary problem consisting of the initial-boundary value problem

$$\begin{cases} v_t = \nabla^2 v + g(h^\pm, v), \quad (\mathbf{x}, t) \in \Omega^\pm(t), \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}), \\ v(\eta(t) - 0, t) = v(\eta(t) + 0, t), \\ \frac{d}{d\nu} v(\eta(t) - 0, t) = \frac{d}{d\nu} v(\eta(t) + 0, t), \\ \eta'(t) = C(v(\eta(t), t), \\ w'(t) = -2w(t) + \left(1 - \left(\frac{\eta(t)}{R}\right)^n\right), \end{cases} \tag{2}$$

where $w =< v >$.

In section 2, a change of variables is given which regularizes problem (2) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we have obtained enough regularity of the solution for an analysis of the bifurcation. In section 3, we show the existence of radially symmetric localized equilibrium solutions for (2) and obtain the linearization of problem (2). In the last section, we find the conditions between a_0 and μ which guarantee the existence of the periodic solutions and the bifurcation of the interface problem as a parameter σ varies in two and three dimensions.

2 Regularized system

We now search for an existence problem of radially symmetric equilibrium solutions of (2) with $|\mathbf{x}| = r$, where the center and the interface are located at the origin and $r = \eta$, respectively. The problem is formulated by (see example [7]) :

$$\begin{cases} v_t = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r}\frac{\partial v}{\partial r} - (\mu+1)v + H(r-\eta) + (\mu-1)w, \ r \in (0,R), \ t > 0 \\ \eta'(t) = C(v(\eta), t)\,, t > 0 \ ; \eta(0) = \eta_0 \\ w'(t) = -2w + 1 - \left(\frac{\eta}{R}\right)^n \ ; w(0) = w_0 \\ v_r(0,t) = 0 = v_r(R,t), \ t > 0. \end{cases} \tag{3}$$

As a first step, we obtain more regularity for the solution by semigroup methods, considering $A := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \mu + 1$ as a densely defined operator with the domain $D(A) = \{v \in H^{2,2}((0,R)) \, : \, \frac{\partial v}{\partial r}v(0,t) = 0 = \frac{\partial v}{\partial r}v(R,t)\}$. In order to use the application of semigroup theory to (3), we chose the space

$$X := L_2((0,R)) \text{ with norm } \|\cdot\|_2 \,.$$

We now define $g : [0,R] \times [0,R] \times \mathbb{C} \to \mathbb{C}$, by

$$g(r,\eta,w) := A^{-1}\Big((H(\cdot - \eta))(r) + (\mu-1)w\Big) \ = \int_\eta^R G(r,y)\,dy + \frac{\mu-1}{\mu+1}w$$

and $\gamma : [0,R] \times \mathbb{C} \to \mathbb{C}$, by

$$\gamma(\eta,w) := g(\eta,\eta,w).$$

Here $G : [0,R]^2 \to \mathbb{R}$ is a Green's function of A satisfying the boundary conditions:

$$G(r,z) = \begin{cases} zI_0(r\sqrt{\mu+1})\ \Big(K_0(z\sqrt{\mu+1}) + \frac{K_1(R\sqrt{\mu+1})}{I_1(R\sqrt{\mu+1})}I_0(z\sqrt{\mu+1})\Big)\ , \ 0 < r < z \\ zI_0(z\sqrt{\mu+1})\ \Big(K_0(r\sqrt{\mu+1}) + \frac{K_1(R\sqrt{\mu+1})}{I_1(R\sqrt{\mu+1})}I_0(r\sqrt{\mu+1})\Big), \ z < r < R, \ (n=2), \end{cases}$$

where I_i and K_i are modified Bessel functions $(i = 0, 1)$. For $n = 3$,

$$G(r,z) = \begin{cases} Y_R\ z\ \frac{\sinh(r\sqrt{\mu+1})}{r\sqrt{\mu+1}}\ \big(R\sqrt{\mu+1}\cosh((R-z)\sqrt{\mu+1}) - \sinh((R-z)\sqrt{\mu+1})\big)\big), \ 0 < r < z \\ Y_R\ z\ \frac{\sinh(z\sqrt{\mu+1})}{r\sqrt{\mu+1}}\ \big(R\sqrt{\mu+1}\cosh((R-r)\sqrt{\mu+1}) - \sinh((R-r)\sqrt{\mu+1})\big)\big), \ z < r < R, \end{cases}$$

where Y_R is given by

$$Y_R = \frac{1}{(\sqrt{\mu+1})(R\sqrt{\mu+1}\cosh(R\sqrt{\mu+1}) - \sinh(R\sqrt{\mu+1})}.$$

Applying the transformation $u(t)(r) = v(r,t) - g(r,\eta(t),w(t))$, we then obtain an equivalent abstract evolution equation of (3) :

$$\begin{cases} \dfrac{d}{dt}(u,\eta,w) + \widetilde{A}(u,\eta,w) = \ F(u,\eta,w) \\ (u,\eta,w)(0) = (u_0(r),\eta_0,w_0), \end{cases} \tag{4}$$

where $\widetilde{A}$ is a 3×3 matrix defined on $D(\widetilde{A}) = D(A)\times(0,R)\ \times\mathbb{C}$ and given by

$$\begin{pmatrix} A & 0 & -(\mu-1) \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The nonlinear forcing term F defined on the set $W := \{(u,\eta,w)\in C^1([0,R])\times(0,R)\times\mathbb{C} : u(\eta)+\gamma(\eta,w)\in\mathbf{I}\}\subset_{\text{open}} C^1([0,R])\times(0,R)\times\mathbb{C}$ as

$$F(u,\eta\,,w\,) = \begin{pmatrix} f_2(u,\eta,w)\cdot\big(f_1(\eta)-\frac{\mu-1}{\mu+1}\frac{n}{R^n}\eta^{n-1}\big) \\ f_2(u,\eta,w) \\ 1-\big(\frac{\eta}{R}\big)^n \end{pmatrix},$$

where $f_1 : (0,R)\to X$, $f_1(\eta)(r) := G(r,\eta)$ and $f_2 : W\to\mathbb{C}$, $f_2(u,\eta,w) := C\big(u(\eta),\eta,w\big)$. We define $\xi(\eta) := \int_\eta^R G(\eta,y)\,dy\,,\ \gamma(\eta,w)=\xi(\eta)+\frac{\mu-1}{\mu+1}w$ and $\chi(u,\eta) := u(\eta)+\xi(\eta)$, then

$$\begin{aligned} C(u(\eta),\eta,w) &= C(\chi(u,\eta),w) \\ &= -\frac{1-2a_0-2\big(\chi(u,\eta)+\frac{\mu-1}{\mu+1}w\big)}{\sigma\sqrt{\big(a_0+\chi(u,\eta)+\frac{\mu-1}{\mu+1}w\big)\big(1-a_0-(\chi(u,\eta)+\frac{\mu-1}{\mu+1}w)\big)}}\,. \end{aligned} \tag{5}$$

Lemma 2.1 *The functions $f_1 : (0,R)\to X,\ f_2 : W\to\mathbb{C}$ and $F : W\to X\times\mathbb{C}\times\mathbb{R}$ are continuously differentiable with derivatives given by*

$$f_1'(\eta) = \tfrac{\partial G}{\partial r}(\cdot,\eta),$$

$$Df_2(u,\eta,w)(\hat{u},\hat{\eta},\hat{w}) = C_\chi(\chi(u,\eta),w)\big(u'(\eta)\hat{\eta}+\hat{u}(\eta)+\xi'(\eta)\hat{\eta}\big)+C_w(\chi(u,\eta),w)\hat{w},$$

$$\begin{aligned} DF(u,\eta,w)(\hat{u},\hat{\eta},\hat{w}) &= f_2(u,\eta,w)\cdot(f_1'(\eta)-\tfrac{\mu-1}{\mu+1}\tfrac{n(n-1)}{R^n}\eta^{n-2},0,0)\,\hat{\eta} \\ &\quad +Df_2(u,\eta,w)(\hat{u},\hat{\eta},\hat{w})\cdot(f_1(\eta)-\tfrac{\mu-1}{\mu+1}\tfrac{n}{R^n}\eta^{n-1},1,0)+(0,0,-\tfrac{n}{R^n}\eta^{n-1})\hat{\eta}\,, \end{aligned}$$

where $C_\chi = \frac{\partial C}{\partial\chi}$ and $C_w = \frac{\partial C}{\partial w}$.

The well posedness of solutions is shown in [2] by applying the semigroup theory using the domains of fractional powers $\alpha\in(3/4,1]$ of A and $\widetilde{A}$ ([4]). Moreover, they determined that $F : W\cap D(\widetilde{A}^\alpha)\to D(\widetilde{A})$ is a continuously differentiable function, where $D(\widetilde{A}^\alpha) = D(A^\alpha)\times(0,R)\ \times\mathbb{C}$.

3 Radially symmetric equilibrium solutions

The steady states are solutions of the following problem :

$$\begin{cases} Au^* = G(\cdot,\eta^*)C(u^*(\eta^*)+\gamma(\eta^*,w^*)) + \frac{\mu-1}{\mu+1}\big(2w^* - (1-(\frac{\eta^*}{R})^n)\big), \\ 0 = C(u^*(\eta^*)+\gamma(\eta^*,w^*)), \\ 0 = -2w^* + 1 - (\frac{\eta^*}{R})^n, \\ u^{*\prime}(0) = 0 = u^{*\prime}(R), \end{cases} \tag{6}$$

for $(u^*,\eta^*,w^*) \in D(\widetilde{A}) \cap W$.

Theorem 3.1 *Suppose that $0 < a_0 < \frac{1}{2}$ and $\mu > 1$. Then the stationary problem of (4) has the only stationary solution (u^*,η^*,w^*) for all $\sigma \neq 0$ with $u^* = 0$, $w^* = \frac{1}{2}\Big(1-(\frac{\eta^*}{R})^n\Big)$ and $\eta^* \in (0,R)$. The linearization of F at $(0,\eta^*,w^*)$ is*

$$DF(0,\eta^*,w^*)(\hat{u},\hat{\eta},\hat{w}) = \begin{pmatrix} \frac{4}{\sigma}\big(\hat{u}(\eta^*) + \gamma_\eta(\eta^*,w^*)\hat{\eta} + \gamma_w(\eta^*,w^*)\hat{w}\big)\cdot G(\cdot,\eta^*) - \frac{\mu-1}{\mu+1}\left(-2\hat{w} - \frac{n}{R^n}\,\eta^{*n-1}\,\hat{\eta}\right) \\ \frac{4}{\sigma}\big(\hat{u}(\eta^*) + \gamma_\eta(\eta^*,w^*)\hat{\eta} + \gamma_w(\eta^*,w^*)\hat{w})\big) \\ -2\hat{w} - \frac{n}{R^n}\,\eta^{*n-1}\,\hat{\eta} \end{pmatrix}.$$

The pair $(0,\eta^*,w^*)$ corresponds to a unique steady state (v^*,η^*,w^*) of (3) for $\sigma \neq 0$ with $v^*(r) = g(r,\eta^*,w^*)$.

Proof: From the system of equations (6), it can be seen that η^* and w^* are solutions of the following equations

$$u^* = 0,\ \ C(0,\eta^*,w^*) = 0 \text{ and } w^* = \tfrac{1}{2}\Big(1-(\tfrac{\eta^*}{R})^n\Big). \tag{7}$$

We only check the existence of η^* in equations (5) and (7). Let

$$\Gamma(\eta) := \tfrac{1}{2} - a_0 - \gamma(\eta,w) = \tfrac{1}{2} - a_0 - \xi(\eta) - \tfrac{\mu-1}{2(\mu+1)}\Big(1-(\tfrac{\eta}{R})^n\Big)$$

then

$$\Gamma'(\eta) = -\xi'(\eta) + \tfrac{\mu-1}{2(\mu+1)}\tfrac{n}{R^n}\eta^{n-1},\ \ \Gamma(0) = \tfrac{1}{2} - a_0 - \xi(0) - \tfrac{\mu-1}{2(\mu+1)} = -a_0 \ \text{ and } \ \Gamma(R) = \tfrac{1}{2} - a_0.$$

Since $\xi'(\eta) < 0$ for $n = 2,3$, $\Gamma'(\eta) > 0$ for $\mu > 1$. Hence, there is a unique $\eta^* \in (0,R)$ for $\mu > 1$.

The formula for $DF(0,\eta^*,w^*)$ follows from Lemma 2.1, the relation $C_\chi(0,\eta^*,w^*) = \frac{4}{\sigma}$ and $C_w(0,\eta^*,w^*) = \frac{4(\mu-1)}{\sigma(\mu+1)}$. The corresponding steady state (v^*,η^*,w^*) for (3) is obtained using the transformation (4) and Theorem 2.1 in [2]. ∎

Definition 3.2 *Suppose that* $0 < a_0 < \frac{1}{2}$ *and* $\mu > 1$. *We define (for* $1 \geq \alpha > 3/4$*) the operator* B *which is a linear operator from* $D(\widetilde{A}^{\alpha})$ *to* $D(\widetilde{A})$ *as*

$$B := \frac{\sigma}{4} DF(0, \eta^*, w^*) .$$

We then define $(0, \eta^*, w^*)$ *to be a* Hopf point *for (4) if there exists an* $\epsilon_0 > 0$ *and a* C^1*-curve*

$$(-\epsilon_0 + \tau^*, \tau^* + \epsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times D(\widetilde{A})_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ *denotes the complexification of the real space* Y*) of eigendata for* $-\widetilde{A} + \tau B$ *such that*

(a) $(-\widetilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\widetilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)}\,\overline{\phi(\tau)}$*;*

(b) $\lambda(\tau^*) = i\beta$ *with* $\beta > 0$*;*

(c) $\mathrm{Re}\,(\lambda) \neq 0$ *for all* λ *in the spectrum of* $(-\widetilde{A} + \tau^* B) \setminus \{\pm i\beta\}$*;*

(d) $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ *(transversality).*

where $\tau = 4/\sigma$.

4 Hopf bifurcation analysis

We shall show that there is a Hopf bifurcation from the curve $\sigma \mapsto (0, \eta^*, w^*)$ of a radially symmetric stationary solution. The linearized eigenvalue problem of (4) is

$$-\widetilde{A}(u, \eta, w) + \tau B(u, \eta, w) = \lambda\, I_3(u, \eta, w),$$

where I_3 is an 3 by 3 identity matrix. This is equivalent to

$$\begin{cases} (A + \lambda)u = \tau\big(u(\eta^*) + \gamma_\eta(\eta^*, w^*)\,\eta + \gamma_w(\eta^*, w^*)\,w\,\big) \cdot G(\cdot, \eta^*) + \frac{\mu-1}{\mu+1}\left(2w + \frac{n}{R^n}(\eta^*)^{n-1}\,\eta\right) \\ \lambda\,\eta = \tau\big(u(\eta^*) + \gamma_\eta(\eta^*, w^*)\,\eta + \gamma_w(\eta^*, w^*)\,w\,\big) \\ \lambda\,w = -2w - \frac{n}{R^n}(\eta^*)^{n-1}\,\eta. \end{cases} \tag{8}$$

Our main theorem is stated as below:

Theorem 4.1 *Under the assumptions of Definition 3.2, the problem (4) and (3) has a unique stationary solution* (u^*, η^*, w^*)*, where* $u^* = 0$ *and* $w^* = \frac{1}{2}\big(1 - \frac{n}{R^n}(\eta^*)^{n-1}\big)$ *and* (v^*, η^*, w^*)*, respectively for all* $\tau > 0$. *Thus, there exists a unique* τ^* *such that the linearization* $-\widetilde{A} + \tau^* B$ *has a purely imaginary pair of eigenvalues* β *with* $0 < \beta < 2\sqrt{\mu + 1}$. *The point* $(0, \eta^*, w^*, \tau^*)$ *is then a Hopf point for (4) and there exists a* C^0*-curve of nontrivial periodic orbits for (4) and (3), bifurcating from* $(0, \eta^*, w^*, \tau^*)$ *and* $(v^*, \eta^*, w^*, \tau^*)$*, respectively.*

In order to prove the above theorem, we will consider the following three theorems. The next lemma is needed to show that the steady state is the only Hopf point.

Lemma 4.2 *Under the assumptions of Definition 3.2, the operator* $-\widetilde{A}+\tau^*B$ *has a unique pair of purely imaginary eigenvalues* $\{\pm i\beta\}$. *Then, the point* $(0,\eta^*,w^*,\tau^*)$ *satisfies the conditions (a),(b) and (c) in Definition 3.2.*

Proof: In the proof, we denote $b_n=\frac{n}{R^n}(\eta^*)^{n-1}$, $n=2,3$. We assume without loss of generality that $\beta>0$, and Φ^* is the (normalized) eigenfunction of $-\widetilde{A}+\tau^*B$ with eigenvalue $i\beta$. We have to show that $(\Phi^*,i\beta)$ can be extended to a C^1-curve $\tau\mapsto(\Phi(\tau),\lambda(\tau))$ of eigendata for $-\widetilde{A}+\tau\,B$ with $\mathrm{Re}(\lambda'(\tau^*))\neq 0$. For this let $\Phi^*:=(\psi_0,\eta_0,w_0)\in D(\widetilde{A})$. First, we note that if $w_0=0$ then $\eta_0=0$ (vice versa) in the last equation of (8). We see that $\eta_0\neq 0$ and $w_0\neq 0$, for otherwise, by (8), $(A+i\beta)\psi_0=i\beta\left(\mu G(\cdot,\eta^*)\,\eta_0-\frac{\mu-1}{\mu+1}\,w_0\right)=0$, which is not possible because A is symmetric. So, without loss of generality, we let $\eta_0=1$. Now, we define

$$E: D(A)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{C}\times\mathbb{R}\longrightarrow X_{\mathbf{C}}\times\mathbb{C}\times\mathbb{C}\,,$$

$$E(u,w\,,\lambda,\tau):=\begin{pmatrix}(A+\lambda)u-\tau(u(\eta^*)+\gamma_\eta+\gamma_w\,w)G(\cdot,\eta^*)-\frac{\mu-1}{\mu+1}(2w+b_n)\\ \\ \lambda-\tau(u(\eta^*)+\gamma_\eta+\gamma_w\,w)\\ \\ \lambda w+2w+b_n\end{pmatrix}, \tag{9}$$

where $\gamma_\eta=\gamma_\eta(\eta^*,w^*)=\xi'(\eta^*)$ and $\gamma_w=\gamma_w(\eta^*,w^*)=\frac{\mu-1}{\mu+1}$. The equation $E(u,w,\lambda,\tau)=0$ is equivalent to λ being an eigenvalue of $-\widetilde{A}+\tau B$ with eigenfunction $(u,1,w)$. By (8), we have $E(\psi_0,w_0,i\beta,\tau^*)=0$ which is equivalent to

$$\begin{cases}(A+i\beta)\psi_0=i\beta(\mu G(\cdot,\eta^*)-\frac{\mu-1}{\mu+1}w_0),\\ i\beta=\tau^*\big(\psi_0(\eta^*)+\gamma_\eta+\gamma_w\,w_0\big),\\ i\beta w_0=-2w_0-b_n,\end{cases} \tag{10}$$

To apply the implicit function theorem to (9), we have to check that E is in C^1 and that

$$D_{(u,w,\lambda)}E(\psi_0,w_0,i\beta,\tau^*)\in L(D(A)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{C},X_{\mathbf{C}}\times\mathbb{C}\times\mathbb{C})\ \text{ is an isomorphism.} \tag{11}$$

In addition, the mapping

$$D_{(u,w,\lambda)}E(\psi_0,w_0,i\beta,\tau^*)(\hat{u},\hat{w},\hat{\lambda})$$

$$=\begin{pmatrix}(A+i\beta)\hat{u}+\hat{\lambda}\psi_0-\tau^*\,(\hat{u}(\eta^*)+\gamma_w\,\hat{w})G(\cdot,\eta^*)-2\frac{\mu-1}{\mu+1}\hat{w}\\ \hat{\lambda}-\tau^*\,(\hat{u}(\eta^*)+\gamma_w\hat{w}\,)\\ \hat{\lambda}\,w_0+i\beta\,\hat{w}+2\hat{w}\end{pmatrix}$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{w}, \hat{\lambda}) \longmapsto \big((A + i\beta)\hat{u},\ \hat{w},\ \hat{\lambda}\big)$$

which is invertible. In order to verify (11), it suffices to show that the system of equations

$$D_{(u,w,\lambda)}E(\psi_0, w_0, i\beta, \tau^*)(\hat{u}, \hat{w}, \hat{\lambda}) = 0$$

which are

$$\begin{cases} (A + i\beta)\hat{u} + \hat{\lambda}\psi_0 = \tau^* \left(\hat{u}(\eta^*) + \gamma_w \hat{w}\right) G(\cdot, \eta^*) + 2\frac{\mu-1}{\mu+1}\hat{w} \\ \hat{\lambda} = \tau^* \left(\hat{u}(\eta^*) + \gamma_w \hat{w}\,\right) \\ \hat{\lambda}\, w_0 = -i\beta\, \hat{w} - 2\hat{w} \end{cases} \tag{12}$$

necessarily implies that $\hat{u} = 0$, $\hat{w} = 0$ and $\hat{\lambda} = 0$. If we define $\phi := \psi_0 - G(\cdot, \eta^*) + \frac{\mu-1}{\mu+1} w_0$, then the first equation of (12) is given by

$$(A + i\beta)\hat{u} + \hat{\lambda}\phi = -i\beta \frac{\mu - 1}{\mu + 1}\hat{w}\,. \tag{13}$$

Since $(v, \eta, w, \lambda) = (\psi_0, 1, w_0, i\beta)$ solves (8), ϕ is a solution to the equation

$$(A + i\,\beta)\phi = -\delta_{\eta^*} + (\mu - 1)w_0, \tag{14}$$

and

$$\begin{cases} \frac{i\beta}{\tau^*} = \phi(\eta^*) + \mu G(\eta^*, \eta^*) - \frac{\mu-1}{\mu+1}w_0 + \gamma_\eta + \gamma_w w_0 = \phi(\eta^*) + \mu\, G(\eta^*, \eta^*) + \xi'(\eta^*) \\ i\beta w_0 = -2w_0 - b_n. \end{cases} \tag{15}$$

Multiplying (14) by r^{n-1} $(n = 2, 3)$ and integrating, we now obtain

$$\int_0^R \Big(-\phi_{rr} - \frac{n-1}{r}\,\phi_r + (\mu + 1 + i\beta)\phi\Big)\, r^{n-1}dr = \int_0^R \Big(-\delta_{\eta^*} + (\mu - 1)w_0\Big) r^{n-1}\, dr$$

which implies that

$$(\mu + 1 + i\beta) \int r^{n-1}\phi = -(\eta^*)^{n-1} + (\mu - 1)w_0\, \frac{R^n}{n}. \tag{16}$$

We now multiply (13) by r^{n-1} and (14) by $r^{n-1}\,\phi$. Next, by integrating the resulting equations, we obtain

$$(\mu + 1 + i\beta) \int r^{n-1}\hat{u} + \hat{\lambda} \int r^{n-1}\,\phi = -i\beta\frac{\mu - 1}{\mu + 1} \cdot \frac{R^n}{n}\,\hat{w} \tag{17}$$

and

$$(\mu+1+i\beta)\int r^{n-1}\,\phi^2 = -(\eta^*)^{n-1}\phi(\eta^*) + (\mu-1)w_0\int r^{n-1}\,\phi. \qquad (18)$$

Multiplying (13) by $r^{n-1}\,\phi$ and (14) by $r^{n-1}\,\hat{u}$ and then subtracting resultants from each other,

$$\hat{\lambda}\int r^{n-1}\,\phi^2 = (\eta^*)^{n-1}\,\hat{u}(\eta^*) - i\beta\frac{\mu-1}{\mu+1}\hat{w}\int r^{n-1}\,\phi - (\mu-1)w_0\int r^{n-1}\,\hat{u}. \qquad (19)$$

Multiplying (19) by $(\mu+1+i\beta)$ and using (16), (17) and (18) in the resulting equation, we obtain

$$\begin{aligned} 0 &= \hat{\lambda}\phi(\eta^*) + (\mu+1+i\beta)\hat{u}(\eta^*) + i\beta\frac{\mu-1}{\mu+1}\hat{w} \\ &= \hat{\lambda}\Big[\frac{\mu+1+2i\beta}{\tau^*} - (\mu G(\eta^*,\eta^*) + \xi'(\eta^*)) - \frac{(\mu-1)\,b_n}{(2+i\beta)^2}\Big]. \end{aligned} \qquad (20)$$

Suppose $\hat{\lambda} \neq 0$, then the real part and the imaginary part of (20) are given by

$$\begin{cases} 0 &= \dfrac{\mu+1}{\tau^*} - (G(\eta^*,\eta^*) + \xi'(\eta^*)) - \dfrac{(\mu-1)\,b_n}{(4+\beta^2)^2}(4-\beta^2) \\ 0 &= 2\beta\Big(\dfrac{1}{\tau^*} + \dfrac{2(\mu-1)\,b_n}{(4+\beta^2)^2}\Big). \end{cases} \qquad (21)$$

This leads to a contradiction due to $\mu > 1$. Hence, we should have $\hat{\lambda} = 0$ and thus $\hat{w} = 0$ and $\hat{u} = 0$. ∎

Theorem 4.3 *Under the assumptions of Definition 3.2,* $(0, \eta^*, w^*, \tau^*)$ *satisfies the transversality condition and hence this is a Hopf point for (4).*

Proof: By implicit differentiation of $E(\psi_0(\tau), w(\tau), \lambda(\tau), \tau) = 0$, we find that

$$D_{(u,w,\lambda)}E(\psi_0, w_0, i\beta, \tau^*)(\psi_0'(\tau^*), w'(\tau^*), \lambda'(\tau^*)) = \begin{pmatrix} \big(\mu\,G(\eta^*,\eta^*) - \frac{\mu-1}{\mu+1}\,b_n\big)(\psi_0(\eta^*) + \gamma_\eta + \gamma_w\,w_0) \\ \psi_0(\eta^*) + \gamma_\eta + \gamma_w\,w_0 \\ 0 \end{pmatrix}.$$

This means that the functions $\tilde{u} := \psi_0'(\tau^*)$, $\tilde{w} := w'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$\begin{cases} (A+i\beta)\tilde{u} + \tilde{\lambda}\psi_0 - \tau^*(\tilde{u}(\eta^*) + \gamma_w\,\tilde{w})(G(\eta^*,\eta^*) - \frac{\mu-1}{\mu+1}\,b_n) - 2\frac{\mu-1}{\mu+1}\tilde{w} \\ \qquad = (\psi_0(\eta^*) + \gamma_\eta + \gamma_w\,w_0)(G(\eta^*,\eta^*) - \frac{\mu-1}{\mu+1}\,b_n) \\ \tilde{\lambda} - \tau^*(\tilde{u}(\eta^*) + \gamma_w\,\tilde{w}) = \psi_0(\eta^*) + \gamma_\eta + \gamma_w\,w_0 \\ \tilde{\lambda}\,w_0 + (2+i\,\beta)\,\tilde{w} = 0\,. \end{cases} \qquad (22)$$

By letting $\phi := \psi_0 - G(\cdot, \eta^*) + \frac{\mu-1}{\mu+1}\, w_0$ as before, then

$$(A + i\beta)\tilde{u} + \tilde{\lambda}\phi = -i\,\beta\,\frac{\mu-1}{\mu+1}\,\tilde{w}\,. \tag{23}$$

From (12) and (22), we obtain

$$\frac{\tilde{\lambda}}{\tau^*} = \tilde{u}(\eta^*) + \gamma_w\,\tilde{w} + \frac{i\beta}{\tau^{*2}}. \tag{24}$$

Multiplying (14) by $\overline{(\mu+1+i\beta)\, r^{n-1}\,\phi}$ and then integrating the resulting equation, we obtain

$$((\mu+1)^2+\beta^2)\int r^{n-1}\,|\phi|^2 = -(\eta^*)^{n-1}\,\overline{(\mu+1+i\beta)\,\phi(\eta^*)} + (\mu-1)w_0\,\overline{(\mu+1+i\beta)\int r^{n-1}\,\phi}$$
$$= (\eta^*)^{n-1}\Big((\mu+1)\big(G(\eta^*,\eta^*)+\xi'(\eta^*)+\tfrac{\beta^2}{\tau^*}\big) + \tfrac{2(\mu-1)\,b_n}{4+\beta^2} + (\mu-1)^2\,|w_0\,|^2\,\tfrac{R^n}{n}\Big)$$
$$+\,i\,(\eta^*)^{n-1}\,\beta\Big(\tfrac{\mu+1}{\tau^*} - G(\eta^*,\eta^*) - \xi'(\eta^*) - \tfrac{(\mu-1)\,b_n}{4+\beta^2}\Big).$$

The imaginary part of the above equation is given by

$$\frac{\mu+1}{\tau^*} - G(\eta^*,\eta^*) - \xi'(\eta^*) - \frac{(\mu-1)\,b_n}{4+\beta^2} = 0. \tag{25}$$

Multiplying (23) by $r^{n-1}\,\phi$ and (14) by $r^{n-1}\,\tilde{u}$ and integrating followed by subtracting the resultants from each other, we get

$$\tilde{\lambda}\int r^{n-1}\,\phi^2 + (\mu-1)w_0\int r^{n-1}\,\tilde{u} + i\beta\tfrac{\mu-1}{\mu+1}\,\tilde{w}\int r^{n-1}\,\phi = (\eta^*)^{n-1}\;\tilde{u}(\eta^*). \tag{26}$$

Multiplying (26) by $(\mu+1+i\beta)$ and using (16), (17) and (18) in the resulting equation, we obtain

$$(\eta^*)^{n-1}\,\tilde{u}(\eta^*)(\mu+1+i\beta) = \tilde{\lambda}\Big(-(\eta^*)^{n-1}\,\phi(\eta^*) + (\mu-1)w_0)\int r^{n-1}\,\phi\Big)$$
$$+(\mu-1)w_0\Big(-\tilde{\lambda}\int r^{n-1}\,\phi - i\beta\tfrac{\mu-1}{\mu+1}\,\tilde{w}\tfrac{R^n}{n}\Big) + i\beta\,\tfrac{\mu-1}{\mu+1}\,\tilde{w}\Big(-(\eta^*)^{n-1} + (\mu-1)w_0\tfrac{R^n}{n}\Big)$$

which implies that

$$0 = \tilde{\lambda}\,\phi(\eta^*) + i\beta\,\tfrac{\mu-1}{\mu+1}\,\tilde{w} + (\mu+1+i\beta)\tilde{u}(\eta^*).$$

Using (15) and (22), we have

$$(\mu+1+i\beta)\frac{i\beta}{\tau^{*2}} = \tilde{\lambda}\Big(\frac{\mu+1+2i\beta}{\tau^*} - (G(\eta^*,\eta^*)+\xi'(\eta^*)) - \frac{(\mu-1)b_n}{(2+i\beta)^2}\Big).$$

The real part of $\widetilde{\lambda}$ is given by

$$\frac{\beta}{\tau^*}\left((\mu+1)^2+\beta^2\right)\,\mathrm{Re}\tilde{\lambda} = |\tilde{\lambda}\,|^2\,((\mu+1)Q-\beta\,P), \tag{27}$$

where

$$P = \tfrac{\mu+1}{\tau^*} - \Big(\mu\,G(\eta^*,\eta^*)+\xi'(\eta^*)\Big) - \tfrac{(\mu-1)b_n}{(4+\beta^2)^2}(4-\beta^2), \quad Q = \tfrac{2\beta}{\tau^*} + \beta\,\tfrac{4(\mu-1)}{(4+\beta^2)^2}$$

and

$$(\mu+1)Q-\beta\,P = \beta\Big(\tfrac{\mu+1}{\tau^*} + G(\eta^*,\eta^*)+\xi'(\eta^*)+\tfrac{(\mu-1)b_n}{(4+\beta^2)^2}(4\mu+8-\beta^2)\Big). \tag{28}$$

Applying (25) to (28), we then arrive at

$$(\mu+1)Q-\beta\,P \;\; = 2\beta\Big(G(\eta^*,\eta^*)+\xi'(\eta^*)+2(\mu+3)\,\tfrac{(\mu-1)b_n}{(4+\beta^2)^2}\Big)$$

which is positive for $\mu > 1$. Therefore, $\mathrm{Re}\lambda'(\tau^*) > 0$ for $\beta > 0$ and $\mu > 1$. Hence, by the Hopf-bifurcation theorem in [2], there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^* for either cases. □

The next theorem shows a critical Hopf point τ^* exists uniquely.

Theorem 4.4 *Under the assumptions of Definition 3.2, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (8) with β, $0<\beta<2\sqrt{\mu+1}$ for a unique critical point $\tau^*>0$ in order for $(0,\eta^*,w^*,\tau^*)$ to be a Hopf point.*

Proof: We only need to show that the function $(u,w,\beta,\tau)\mapsto E(u,w,i\beta,\tau)$ has a unique zero with $\beta>0$ and $\tau>0$. This means solving the systems of equations (8) with $\lambda = i\beta\,, \eta_0 = 1$ and $\psi_0 = \phi+\mu\,G(\cdot,\eta^*) - \frac{\mu-1}{\mu+1}\,w_0\,,$

$$\begin{cases} (A+i\beta)\phi = -\delta_{\eta^*}+(\mu-1)w_0, \\ \frac{i\beta}{\tau^*} \; = \phi(\eta^*)+G(\eta^*,\eta^*)+\xi'(\eta^*), \\ i\beta\,w_0 = -2w_0-b_n\,. \end{cases} \tag{29}$$

From the second equation,

$$\frac{i\beta}{\tau^*} \; = -G_\beta(\eta^*,\eta^*)+\frac{\mu-1}{\mu+1+i\beta}w_0+G(\eta^*,\eta^*)+\xi'(\eta^*), \tag{30}$$

where G_β is a Green's function of the differential operator $A+i\beta$. The real and imaginary parts of this above equation are given by

$$\begin{cases} \frac{\beta}{\tau^*} = -\mathrm{Im}G_\beta(\eta^*,\eta^*)+\frac{\beta\,b_n(\mu-1)(\mu+3)}{(4+\beta^2)((\mu+1)^2+\beta^2)}, \\ 0 = -\mathrm{Re}G_\beta(\mu^*,\mu^*)+G(\eta^*,\eta^*)+\xi'(\eta^*)-(\mu-1)\,b_n\,\frac{2(\mu+1)-\beta^2}{(4+\beta^2)((\mu+1)^2+\beta^2)}. \end{cases}$$

Since $\mathrm{Im}G_\beta(\eta^*,\eta^*) < 0$ (Lemma 12 ([2])), there is a critical point τ^* given the existence of β for $\mu > 1$. We now define

$$T(\beta) := -\mathrm{Re}G_\beta(\mu^*,\mu^*) + G(\eta^*,\eta^*) + \xi'(\eta^*) - (\mu-1)\, b_n \frac{2(\mu+1)-\beta^2}{(4+\beta^2)((\mu+1)^2+\beta^2)}$$

then $T(\infty) = G(\eta^*,\eta^*) + \xi'(\eta^*) > 0$ and $T(0) = \xi'(\eta^*) - \frac{(\mu-1)b_n}{2\mu(\mu+1)} < 0$ for $\mu > 1$. If we show that $T'(\beta) < 0$, then the existence of β is proved.

$$\begin{aligned} T'(\beta) \; &= -(\mathrm{Re}G_\beta(\eta^*,\eta^*))' \\ &\quad + \frac{2\beta\, b_n\,(\mu-1)}{(4+\beta^2)^2((\mu+1)^2+\beta^2)^2}\Big(\beta^2(4(\mu+1)-\beta^2) + 2(\mu+1)(\mu^2+4\mu+7)\Big). \end{aligned}$$

Since $(\mathrm{Re}G_\beta(\eta^*,\eta^*))' < 0$ (Lemma 3.2 ([2])), we have $T'(\beta) > 0$ for all $\beta^2 < 4(\mu+1)$ and $\mu > 1$. ∎

There is a unique pure imaginary eigenvalue β with $0 < \beta < 2\sqrt{\mu+1}$ and the critical point τ^* of (3) for $\mu > 1$. Thus, there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^* under the condition of Theorem 4.1. Also, this shows the difference from the result in [3]: The Hopf bifurcation for the interface problem, in which the activator is inhibited by an activator as well as the spatial average of its activator, occurs for $0 < \mu < 1$.

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