Applied Mathematical Sciences, Vol. 6, 2012, no. 126, 6273 - 6280

Generalized Fuzzy Cosets and Radicals

Young Bae Jun

Department of Mathematics Education
Gyeongsang National University, Chinju 660-701, Korea
skywine@gmail.com

Kyoung Ja Lee*

Department of Mathematics Education
Hannam University, Daejeon 306-791, Korea
lsj1109@hotmail.com

Mehmet Ali Öztürk

Department of Mathematics
Adiyaman University, 02040 Adiyaman, Turkey
mehaliozturk@gmail.com

Abstract

A generalization of an $(\in, \in\vee q)$-fuzzy coset and an $(\in, \in\vee q)$-fuzzy radical is discussed. A ring related to the notion of $(\in, \in\vee q_k)$-fuzzy cosets is established. It is proved that the $(\in, \in\vee q_k)$-fuzzy radical of an $(\in, \in\vee q_k)$-fuzzy ideal is also an $(\in, \in\vee q_k)$-fuzzy ideal.

Mathematics Subject Classification: 16N99, 08A72

Keywords: Fuzzy subring (ideal), $(\in, \in\vee q)$-fuzzy subring (ideal), $(\in, \in\vee q_k)$-fuzzy subring (ideal), $(\in, \in\vee q)$-fuzzy coset, $(\in, \in\vee q_k)$-fuzzy coset, $(\in, \in\vee q)$-fuzzy radical, $(\in, \in\vee q_k)$-fuzzy radical

1 Introduction

In 1996, Bhakat and Das [1] defined $(\in, \in\vee q)$-fuzzy subrings and ideals of a ring. Also, they introduced the concepts of $(\in, \in\vee q)$-fuzzy semiprime, prime, semiprimary, primary and maximal ideals, and obtained characterization of

*Corresponding author.
This paper has been supported by the 2012 Hannam University Research Fund.

such fuzzy ideals. Jun et al. [2] used more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set to obtain generalizations of results in the paper [1]. Bhakat and Das [1] discussed $(\in, \in\vee q)$-fuzzy cosets determined by $(\in, \in\vee q)$-fuzzy ideals and $(\in, \in\vee q)$-fuzzy radicals of $(\in, \in\vee q)$-fuzzy ideals. In this paper, we try to have more general form of an $(\in, \in\vee q)$-fuzzy coset and an $(\in, \in\vee q)$-fuzzy radical. We introduce the notion of $(\in, \in\vee q_k)$-fuzzy cosets determined by $(\in, \in\vee q_k)$-fuzzy ideals and $(\in, \in\vee q_k)$-fuzzy radicals of $(\in, \in\vee q_k)$-fuzzy ideals. The important achievement of the study with an $(\in, \in\vee q_k)$-fuzzy coset and an $(\in, \in\vee q_k)$-fuzzy radical is that the notion of an $(\in, \in\vee q)$-fuzzy coset and an $(\in, \in\vee q)$-fuzzy radical is a special case of an $(\in, \in\vee q_k)$-fuzzy coset and an $(\in, \in\vee q_k)$-fuzzy radical, and thus so many results in the paper [1] are corollaries of our results obtained in this paper.

2 Preliminaries

Definition 2.1 ([3]). A fuzzy subset μ of a ring R is called a fuzzy subring of R if it satisfies the following conditions:

(a1) $(\forall x, y \in R)$ $(\mu(x-y) \geq \min\{\mu(x), \mu(y)\})$,

(a2) $(\forall x, y \in R)$ $(\mu(xy) \geq \min\{\mu(x), \mu(y)\})$,

Definition 2.2 ([3]). A fuzzy subset μ of a ring R is called a fuzzy left (resp. right) ideal of R if it satisfies (a1) and

(a3) $(\forall x, y \in R)$ $(\mu(xy) \geq \mu(y))$ (resp. $\mu(xy) \geq \mu(x)$).

If μ is both a fuzzy left and fuzzy right ideal of R, then μ is said to be a fuzzy ideal of R.

For any fuzzy subset μ of a set X and any $t \in [0, 1]$ the set $U(\mu; t) = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of μ. A fuzzy subset μ of a set X of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by $[x; t]$.

For a fuzzy subset μ of a set X, we say that a fuzzy point $[x; t]$ is

(a4) contained in μ, denoted by $[x; t] \in \mu$, ([4]) if $\mu(x) \geq t$.

(a5) quasi-coincident with μ, denoted by $[x; t]\, q\, \mu$, ([4]) if $\mu(x) + t > 1$.

For a fuzzy point $[x; t]$ and a fuzzy subset μ of a set X, we say that (a6) $[x;t] \in\vee q\ \mu$ if $[x;t] \in \mu$ or $[x;t]\, q\, \mu$.

(a7) $[x;t]\, \overline{\alpha}\, \mu$ if $[x;t]\, \alpha\, \mu$ does not hold for $\alpha \in \{\in, q, \in\vee q\}$.

Definition 2.3 ([1]). A fuzzy subset μ of a ring R is called an $(\in, \in\vee q)$-fuzzy subring of R if for any $x, y \in R$ and $t, r \in (0,1]$,

(a8) $[x;t] \in \mu,\ [y;r] \in \mu \Rightarrow [x+y; \min\{t,r\}] \in\vee q\, \mu$,

(a9) $[x;t] \in \mu \Rightarrow [-x;t] \in\vee q\, \mu$,

(a10) $[x;t] \in \mu,\ [y;r] \in \mu \Rightarrow [xy; \min\{t,r\}] \in\vee q\, \mu$.

Definition 2.4 ([1]). A fuzzy subset μ of a ring R is called an $(\in, \in\vee q)$-fuzzy ideal of R if

(a11) μ is an $(\in, \in\vee q)$-fuzzy subring of R,

(a12) $(\forall x, y \in R)\ (\forall t \in (0,1])\ ([x;t] \in \mu \Rightarrow [xy;t] \in\vee q\, \mu,\ [yx;t] \in\vee q\, \mu)$.

In what follows, let R denote a ring and k an arbitrary element of $[0,1)$ unless otherwise specified. For a fuzzy point $[x;t]$ and a fuzzy subset μ of R, we say that

(b1) $[x;t]\ q_k\ \mu$ if $\mu(x) + t + k > 1$.

(b2) $[x;t] \in\vee q_k\ \mu$ if $[x;t] \in \mu$ or $[x;t]\ q_k\ \mu$.

(b3) $[x;t]\, \overline{\alpha}\, \mu$ if $[x;t]\, \alpha\, \mu$ does not hold for $\alpha \in \{q_k, \in\vee q_k\}$.

Definition 2.5 ([2]). A fuzzy subset μ of R is called an $(\in, \in\vee q_k)$-fuzzy subring of R if for any $x, y \in R$ and $t, r \in (0,1]$,

(b4) $[x;t] \in \mu,\ [y;r] \in \mu \Rightarrow [x+y; \min\{t,r\}] \in\vee q_k\, \mu$,

(b5) $[x;t] \in \mu \Rightarrow [-x;t] \in\vee q_k\, \mu$,

(b6) $[x;t] \in \mu,\ [y;r] \in \mu \Rightarrow [xy; \min\{t,r\}] \in\vee q_k\, \mu$.

Lemma 2.6 ([2]). *Condition* (b4) *is equivalent to*

$$(\forall x, y \in R)\ (\mu(x+y) \geq \min\{\mu(x), \mu(y), \tfrac{1-k}{2}\}), \tag{2.1}$$

and condition (b5) *is equivalent to*

$$(\forall x \in R)\ (\mu(-x) \geq \min\{\mu(x), \tfrac{1-k}{2}\}). \tag{2.2}$$

Definition 2.7 ([2]). A fuzzy subset μ of R is called an $(\in, \in\vee q_k)$-fuzzy ideal of R if it is an $(\in, \in\vee q_k)$-fuzzy subring of R satisfying the following condition (b7) $(\forall x, y \in R)$ $(\forall t \in (0,1])$ $([x;t] \in \mu \Rightarrow [xy;t] \in\vee q_k\, \mu,\ [yx;t] \in\vee q_k\, \mu)$.

It is easy to check that the condition (b7) is equivalent to the following condition

$$(\forall x, y \in R)\,(\min\{\mu(xy), \mu(yx)\} \geq \min\{\mu(x), \tfrac{1-k}{2}\}). \tag{2.3}$$

Hence we have the following characterization of an $(\in, \in\vee q_k)$-fuzzy ideal of R.

Theorem 2.8 ([2]). *A fuzzy subset μ of R is an $(\in, \in\vee q_k)$-fuzzy ideal of R if and only if it satisfies* (2.3) *and*

$$(\forall x, y \in R)\,(\mu(x - y) \geq \min\{\mu(x), \mu(y), \tfrac{1-k}{2}\}). \tag{2.4}$$

3 $(\in, \in\vee q_k)$-fuzzy cosets

Definition 3.1. Let μ be an $(\in, \in\vee q_k)$-fuzzy ideal of R and $x \in R$. The fuzzy subset μ_x of R defined by $\mu_x(u) = \min\{\mu(u - x), \frac{1-k}{2}\}$ for all $u \in R$ is called the $(\in, \in\vee q_k)$-fuzzy coset determined by x and μ.

The $(\in, \in\vee q_k)$-fuzzy coset determined by x and μ with $k = 0$ is called the $(\in, \in\vee q)$-fuzzy coset determined by x and μ.

Denote by R^*_μ (resp. R_μ) the set of all $(\in, \in\vee q_k)$-fuzzy cosets (resp. $(\in, \in\vee q)$-fuzzy cosets) of μ in R.

Theorem 3.2. *Let μ be an $(\in, \in\vee q_k)$-fuzzy ideal of R. Then R^*_μ is a ring under the addition $(+)$ and multiplication $(\cdot)$ defined as follows:*

$$\mu_x + \mu_y = \mu_{x+y} \text{ and } \mu_x \cdot \mu_y = \mu_{xy} \text{ for all } \mu_x, \mu_y \in R^*_\mu.$$

Proof. Let $a, b, c, d \in R$ be such that $\mu_a = \mu_b$ and $\mu_c = \mu_d$. Then

$$\min\left\{\mu(r - a), \tfrac{1-k}{2}\right\} = \min\left\{\mu(r - b), \tfrac{1-k}{2}\right\} \tag{3.1}$$

and

$$\min\left\{\mu(r - c), \tfrac{1-k}{2}\right\} = \min\left\{\mu(r - d), \tfrac{1-k}{2}\right\} \tag{3.2}$$

for all $r \in R$. If we take $r = a$ and $r = c$ in (3.1) and (3.2) respectively, then

$$\min\left\{\mu(a - b), \tfrac{1-k}{2}\right\} = \min\left\{\mu(0), \tfrac{1-k}{2}\right\} = \tfrac{1-k}{2} \tag{3.3}$$

and

$$\min\left\{\mu(c - d), \tfrac{1-k}{2}\right\} = \min\left\{\mu(0), \tfrac{1-k}{2}\right\} = \tfrac{1-k}{2}. \tag{3.4}$$

If we take $r = a + c - d$ in (3.1), then $\min\left\{\mu(a+c-d-b), \frac{1-k}{2}\right\} = \min\left\{\mu(c-d), \frac{1-k}{2}\right\} = \frac{1-k}{2}$ and so $\mu(a+c-d-b) \geq \frac{1-k}{2}$. For every $r \in R$, we have

$$\begin{aligned}(\mu_a + \mu_c)(r) = \mu_{a+c}(r) &= \min\left\{\mu(r-(a+c)), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu((r-b-d)-(a+c-b-d)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(r-b-d), \mu(a+c-b-d), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(r-b-d), \tfrac{1-k}{2}\right\} = \mu_{b+d}(r) = (\mu_b + \mu_d)(r).\end{aligned}$$

Hence $\mu_a + \mu_c \geq \mu_b + \mu_d$. Similarly, $\mu_b + \mu_d \geq \mu_a + \mu_c$, and so $\mu_a + \mu_c = \mu_b + \mu_d$. Therefore the addition "+" is well defined. Now, for any $r \in R$, we get

$$\begin{aligned}(\mu_a \cdot \mu_c)(r) = \mu_{ac}(r) &= \min\left\{\mu(r-ac), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu((r-bd)-(ac-bd)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(r-bd), \mu(ac-bd), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(r-bd), \mu((a-b)c-b(d-c)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(r-bd), \mu((a-b)c), \mu(b(d-c)), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(r-bd), \mu(a-b), \mu(c-d), \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mu(r-bd), \tfrac{1-k}{2}\right\} = \mu_{bd}(r) = (\mu_b \cdot \mu_d)(r).\end{aligned}$$

Hence $\mu_a \cdot \mu_c \geq \mu_b \cdot \mu_d$. Similarly, $\mu_b \cdot \mu_d \geq \mu_a \cdot \mu_c$. Therefore $\mu_a \cdot \mu_c = \mu_b \cdot \mu_d$, that is, the multiplication "$\cdot$" is well defined. It is now easy to prove that R^*_μ is a ring with μ_0 as the null element and μ_{-x} is the negative of μ_x. □

Corollary 3.3 ([1]). *Let μ be an $(\in, \in\vee q)$-fuzzy ideal of R. Then R_μ is a ring under the addition $(+)$ and multiplication $(\cdot)$ defined as follows:*

$$\mu_x + \mu_y = \mu_{x+y} \text{ and } \mu_x \cdot \mu_y = \mu_{xy} \text{ for all } \mu_x, \mu_y \in R_\mu.$$

Theorem 3.4. *For any $(\in, \in\vee q_k)$-fuzzy ideal μ of R, we define the fuzzy subset $\overline{\mu}$ of R^*_μ by $\overline{\mu}(\mu_x) = \mu(x)$ for all $\mu_x \in R^*_\mu$ where $x \in R$. Then $\overline{\mu}$ is an $(\in, \in\vee q_k)$-fuzzy ideal of R^*_μ.*

Proof. Let $x, y \in R$. Then $\overline{\mu}(\mu_x - \mu_y) = \overline{\mu}(\mu_{x-y}) = \mu(x-y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} = \min\left\{\overline{\mu}(\mu_x), \overline{\mu}(\mu_y), \frac{1-k}{2}\right\}$ and $\min\{\overline{\mu}(\mu_x \cdot \mu_y), \overline{\mu}(\mu_y \cdot \mu_x)\} = \min\{\overline{\mu}(\mu_{xy}), \overline{\mu}(\mu_{yx})\} = \min\{\mu(xy), \mu(yx)\} \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} = \min\left\{\overline{\mu}(\mu_x), \frac{1-k}{2}\right\}$.

Using Theorem 2.8, $\overline{\mu}$ is an $(\in, \in\vee q_k)$-fuzzy ideal of R^*_μ. □

Corollary 3.5 ([1]). *For any $(\in, \in\vee q)$-fuzzy ideal μ of R, we define the fuzzy subset $\overline{\mu}$ of R_μ by $\overline{\mu}(\mu_x) = \mu(x)$ for all $\mu_x \in R_\mu$ where $x \in R$. Then $\overline{\mu}$ is an $(\in, \in\vee q)$-fuzzy ideal of R_μ.*

Lemma 3.6. *For any $(\in, \in\vee q_k)$-fuzzy ideal μ of R, let $A := \{x \in R \mid \mu_x \geq \mu_0\}$ and $B := \{x \in R \mid \mu_x = \mu_0\}$. Then $A = B = U\left(\mu; \frac{1-k}{2}\right)$.*

Proof. If $x \in A$, then $\mu_x(r) \geq \mu_0(r)$ for all $r \in R$, and so $\mu(x) \geq \frac{1-k}{2}$ by taking $r = 0$. It follows that $\mu_0(r) = \min\left\{\mu(r), \frac{1-k}{2}\right\} = \min\left\{\mu(r-x+x), \frac{1-k}{2}\right\} \geq \min\left\{\mu(r-x), \mu(x), \frac{1-k}{2}\right\} = \min\left\{\mu(r-x), \frac{1-k}{2}\right\} = \mu_x(r)$ for all $r \in R$, that is, $\mu_0 \geq \mu_x$. Hence $\mu_0 = \mu_x$, i.e., $x \in B$. Thus $A = B$. Next, if $x \in A$ then $\mu_x(r) \geq \mu_0(r)$ for all $r \in R$. Thus $\mu(-x) \geq \min\left\{\mu(0), \frac{1-k}{2}\right\}$. Again, $\mu(x) \geq \min\left\{\mu(-x), \frac{1-k}{2}\right\}$. Hence $\mu(x) \geq \frac{1-k}{2}$, and so $x \in U\left(\mu; \frac{1-k}{2}\right)$. This shows that $A \subseteq U\left(\mu; \frac{1-k}{2}\right)$. Now let $x \in U\left(\mu; \frac{1-k}{2}\right)$. Then $\mu(x) \geq \frac{1-k}{2}$, which implies that $\mu(r-x) \geq \min\left\{\mu(r), \mu(x), \frac{1-k}{2}\right\} = \min\left\{\mu(r), \frac{1-k}{2}\right\}$ for all $r \in R$. Therefore $\mu_x(r) = \min\left\{\mu(r-x), \frac{1-k}{2}\right\} \geq \min\left\{\mu(r), \frac{1-k}{2}\right\} = \mu_0(r)$ for all $r \in R$, i.e., $\mu_x \geq \mu_0$. Thus $U\left(\mu; \frac{1-k}{2}\right) \subseteq A$, and consequently $A = U\left(\mu; \frac{1-k}{2}\right)$. □

Corollary 3.7 ([1]). *For any $(\in, \in\vee q)$-fuzzy ideal μ of R, let $A = \{x \in R \mid \mu_x \geq \mu_0\}$ and $B = \{x \in R \mid \mu_x = \mu_0\}$. Then $A = B = U(\mu; 0.5)$.*

Theorem 3.8. *If μ is an $(\in, \in\vee q_k)$-fuzzy ideal of R, then the mapping*

$$f : R \to R^*_\mu,\ x \mapsto \mu_x$$

is a homomorphism and $ker(f) = U\left(\mu; \frac{1-k}{2}\right)$.

Proof. For any $x, y \in R$, we have $f(x+y) = \mu_{x+y} = \mu_x + \mu_y = f(x) + f(y)$ and $f(xy) = \mu_{xy} = \mu_x \cdot \mu_y = f(x) \cdot f(y)$. Hence f is a homomorphism. Now, $Ker(f) = \{x \in R \mid f(x) = f(0)\} = \{x \in R \mid \mu_x = \mu_0\} = U\left(\mu; \frac{1-k}{2}\right)$ by Lemma 3.6. □

Note that the mapping f in Theorem 3.8 is clearly onto. Hence the first isomorphism theorem induces that $R/U\left(\mu; \frac{1-k}{2}\right)$ is isomorphic to R^*_μ.

Corollary 3.9 ([1]). *If μ is an $(\in, \in\vee q)$-fuzzy ideal of R, then the mapping*

$$f : R \to R_\mu,\ x \mapsto \mu_x$$

is a homomorphism and $ker(f) = U(\mu; 0.5)$. Moreover $R/U(\mu; 0.5)$ is isomorphic to R_μ.

Theorem 3.10. *Let μ and $\overline{\mu}$ be $(\in, \in\vee q_k)$-fuzzy ideals of R and R^*_μ, respectively. Define a fuzzy subset ν of R by $\nu(x) = \overline{\mu}(\mu_x)$ for all $x \in R$. Then ν is an $(\in, \in\vee q_k)$-fuzzy ideal of R.*

Proof. For any $x, y \in R$, we have $\min\{\nu(xy), \nu(yx)\} = \min\{\overline{\mu}(\mu_{xy}), \overline{\mu}(\mu_{yx})\} = \min\{\overline{\mu}(\mu_x \cdot \mu_y), \overline{\mu}(\mu_y \cdot \mu_x)\} \geq \min\left\{\overline{\mu}(\mu_x), \frac{1-k}{2}\right\} = \min\left\{\nu(x), \frac{1-k}{2}\right\}$ and $\nu(x-y) = \overline{\mu}(\mu_{x-y}) = \overline{\mu}(\mu_x - \mu_y) \geq \min\left\{\overline{\mu}(\mu_x), \overline{\mu}(\mu_y), \frac{1-k}{2}\right\} = \min\left\{\nu(x), \nu(y), \frac{1-k}{2}\right\}$. Using Theorem 2.8, we conclude that ν is an $(\in, \in\vee q_k)$-fuzzy ideal of R. □

Corollary 3.11 ([1]). *Let μ and $\overline{\mu}$ be $(\in, \in\vee q)$-fuzzy ideals of R and R_μ, respectively. Define a fuzzy subset ν of R by $\nu(x) = \overline{\mu}(\mu_x)$ for all $x \in R$. Then ν is an $(\in, \in\vee q)$-fuzzy ideal of R.*

4 $(\in, \in\vee q_k)$-fuzzy radicals

Definition 4.1. Let μ be an $(\in, \in\vee q_k)$-fuzzy ideal of a commutative ring R. The fuzzy subset $\mathrm{Rad}\mu$ of R defined by

$$\mathrm{Rad}\mu(x) := \begin{cases} \min\left\{\sup\{\mu(x^n) \mid n \in \mathbb{N}\}, \frac{1-k}{2}\right\} & \text{if } \mu(x) < \frac{1-k}{2}, \\ \mu(x) & \text{if } \mu(x) \geq \frac{1-k}{2} \end{cases} \tag{4.1}$$

is called the $(\in, \in\vee q_k)$-fuzzy radical of μ.

The $(\in, \in\vee q_k)$-fuzzy radical of μ with $k = 0$ is called the $(\in, \in\vee q)$-fuzzy radical of μ (see [1]).

Lemma 4.2. *If μ is an $(\in, \in\vee q_k)$-fuzzy ideal of a commutative ring R, then*

(1) $(\forall m \in \mathbb{N})\ (\forall x \in R)\ \left(\mu(mx) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\}\right).$

(2) $(\forall m, n \in \mathbb{N})\ (\forall x \in R)\ \left(m \geq n \Rightarrow \mu(x^m) \geq \min\left\{\mu(x^n), \frac{1-k}{2}\right\}\right).$

Proof. (1) Using (2.1), we have $\mu(2x) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\}$ for all $x \in R$. Thus (1) is true for $m = 2$. Assume that (1) is true for $m = r$. Then $\mu((r+1)x) \geq \min\left\{\mu(rx), \mu(x), \frac{1-k}{2}\right\} \geq \min\left\{\mu(x), \frac{1-k}{2}\right\}$. Thus, by the Mathematical Induction, (1) is valid.

(2) Let $m, n \in \mathbb{N}$ be such that $m \geq n$. Then $\mu(x^m) = \mu(x^{m-n}x^n) \geq \min\left\{\mu(x^n), \frac{1-k}{2}\right\}$ for all $x \in R$. □

If we take $k = 0$ in Lemma 4.2, then we have the following corollary.

Corollary 4.3 ([1])**.** *If μ is an $(\in, \in\vee q)$-fuzzy ideal of a commutative ring R, then*

(1) $(\forall m \in \mathbb{N})\ (\forall x \in R)\ (\mu(mx) \geq \min\{\mu(x), 0.5\}).$

(2) $(\forall m, n \in \mathbb{N})\ (\forall x \in R)\ (m \geq n \Rightarrow \mu(x^m) \geq \min\{\mu(x^n), 0.5\}).$

Theorem 4.4. *If μ is an $(\in, \in\vee q_k)$-fuzzy ideal of a commutative ring R, then its $(\in, \in\vee q_k)$-fuzzy radical is also an $(\in, \in\vee q_k)$-fuzzy ideal of R.*

Proof. For any $x, y \in R$, either $\mu(x-y) < \frac{1-k}{2}$ or $\mu(x-y) \geq \frac{1-k}{2}$. Assume that $\mu(x-y) < \frac{1-k}{2}$. Since R is commutative, we have $(x-y)^{m+n} = ax^m + by^n$ where $a, b \in R$ and $m, n \in \mathbb{N}$. Thus

$$\begin{aligned} \mathrm{Rad}\mu(x-y) &= \min\left\{\sup\left\{\mu((x-y)^r) \mid r \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\} \\ &\geq \sup\left\{\min\left\{\mu((x-y)^r), \tfrac{1-k}{2}\right\} \mid r \in \mathbb{N}\right\} \\ &\geq \min\left\{\mu((x-y)^{m+n}), \tfrac{1-k}{2}\right\} = \min\left\{\mu(ax^m + by^n), \tfrac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(ax^m), \mu(by^n), \tfrac{1-k}{2}\right\} \geq \min\left\{\mu(x^m), \mu(y^n), \tfrac{1-k}{2}\right\}. \end{aligned} \tag{4.2}$$

Since $\mu(x-y) < \frac{1-k}{2}$, we can consider the following three cases:

(i) $\mu(x) < \frac{1-k}{2}$ and $\mu(y) < \frac{1-k}{2}$,

(ii) $\mu(x) \geq \frac{1-k}{2}$ and $\mu(y) < \frac{1-k}{2}$,

(iii) $\mu(x) < \frac{1-k}{2}$ and $\mu(y) \geq \frac{1-k}{2}$.

For the first case, it follows from (4.2) that

$$\begin{aligned}\mathrm{Rad}\mu(x-y) \geq \min\Big\{ &\min\left\{\sup\left\{\mu(x^m) \mid m \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\}, \\ &\min\left\{\sup\left\{\mu(y^n) \mid n \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\}, \tfrac{1-k}{2}\Big\} \\ = &\min\left\{\mathrm{Rad}\mu(x), \mathrm{Rad}\mu(y), \tfrac{1-k}{2}\right\}.\end{aligned}$$

The second case implies that $\mathrm{Rad}\mu(x) = \mu(x)$. Using (4.2), we have

$$\begin{aligned}\mathrm{Rad}\mu(x-y) &\geq \min\left\{\mu(x), \min\left\{\sup\left\{\mu(y^n) \mid n \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\}, \tfrac{1-k}{2}\right\} \\ &= \min\left\{\mathrm{Rad}\mu(x), \mathrm{Rad}\mu(y), \tfrac{1-k}{2}\right\}.\end{aligned}$$

The third case is similar to the second case. Suppose that $\mu(x-y) \geq \frac{1-k}{2}$. Then $\mu(x) < \frac{1-k}{2}$ and $\mu(y) < \frac{1-k}{2}$. Hence $\mathrm{Rad}\mu(x-y) = \mu(x-y) \geq \frac{1-k}{2} \geq \min\left\{\mathrm{Rad}\mu(x), \mathrm{Rad}\mu(y), \frac{1-k}{2}\right\}$. Now, if $\mu(xy) < \frac{1-k}{2}$ then

$$\begin{aligned}&\mathrm{Rad}\mu(xy) = \min\left\{\sup\left\{\mu((xy)^n) \mid n \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\} \\ &= \min\left\{\sup\left\{\mu(x^n y^n) \mid n \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\} \geq \min\left\{\sup\left\{\min\left\{\mu(x^n), \tfrac{1-k}{2}\right\} \mid n \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\} \\ &= \min\left\{\min\left\{\sup\left\{\mu(x^n) \mid n \in \mathbb{N}\right\}, \tfrac{1-k}{2}\right\}, \tfrac{1-k}{2}\right\} = \min\left\{\mathrm{Rad}\mu(x), \tfrac{1-k}{2}\right\}.\end{aligned}$$

Finally assume that $\mu(xy) \geq \frac{1-k}{2}$. Then $\mathrm{Rad}\mu(xy) = \mu(xy) \geq \frac{1-k}{2} \geq \min\left\{\mathrm{Rad}\mu(x), \frac{1-k}{2}\right\}$. Therefore $\mathrm{Rad}\mu$ is an $(\in, \in\vee\, q_k)$-fuzzy ideal of R. □

Corollary 4.5 ([1]). *If μ is an $(\in, \in\vee\, q)$-fuzzy ideal of a commutative ring R, then its $(\in, \in\vee\, q)$-fuzzy radical is also an $(\in, \in\vee\, q)$-fuzzy ideal of R.*

References

[1] S. K. Bhakat, P. Das, Fuzzy subrings and ideals redefined, *Fuzzy Sets and Systems* **81** (1996), 383–393.

[2] Y. B. Jun, M. A. Öztürk and E. H. Roh, General types of fuzzy subrings and ideals, *Comput. Math. Appl.* (submitted).

[3] W. J. Liu, Fuzzy invariant subgroups and fuzzy ideals, *Fuzzy Sets and Systems* **8** (1982), 133-139.

[4] P. M. Pu, Y. M. Liu, Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.* **76** (1980), 571–599.

Received: July, 2012