Applied Mathematical Sciences, Vol. 6, 2012, no. 126, 6281 - 6286

Legendre Wavelets Based Approximation Method for Cauchy Problems

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Abstract

In this paper, we study the Legendre wavelets based approximation method for the solution of Cauchy problems. The properties of Legendre wavelets are used to reduce the problem to the solution of algebraic equations.The function approximation has been chosen in such a way that the connection coefficients can be identified easily and it converges to the exact solution. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique.

Mathematics Subject Classification: 35E15, 35F05, 35F10

Keywords: Legendre Polynomials, Legendre wavelets, Cauchy problems, Legendre wavelet method

1 Introduction

In this paper, we consider the Cauchy problem of first order partial differential equation (PDE)

$$u_t\left(x,t\right)+a\left(x,t\right)u_x\left(x,t\right)=f\left(x\right),x\in R,t>0 \tag{1}$$

$$u\left(x,0\right)=\psi\left(x\right),x\in R. \tag{2}$$

where $a(x,t)=a$ is a constant and $f\left(x\right)=0$. The Cauchy problem plays an important role in theoretical physics. Eq.(1) is a nonlinear equation called the transport equation which can describe many interesting phenomena such as the spread of AIDS, the moving of wind [9]. When $a(x,t)=u(x,t)$, the equation is called the inviscid Burgers' equation arising in one-dimensional stream of particle or fluid having zero viscosity[9]. Xin-Wei Zhou et al [9] applied variational iterational method for Cauchy problems. Lin Jin [4] had studied Cauchy problem by using homotopy perturbation method and Mehdi Gholami Porshokouhi et al [3] had discussed the analytic solution of Cauchy problem. Wavelets are one of the tools to handle the nonlinearity, to provide the exact or approximation solution. In recent years, wavelets have found their place in many applications such as signal processing, image processing, and solution of many differential and integral equations. The main characteristic of wavelets is its ability to convert the given differential and integral equations to a system of linear or nonlinear algebraic equations. Many authors have handled Legendre wavelets by using operational matrices and without using operational matrices for the solution of varieties of differential and integral equations. Recently, S.G.Venkatesh et al [6-8] applied Legendre wavelets for the solution of initial value problems of Bratu-type and higher order Volterra integro-differential equations and they have also discussed theoretical analysis of Legendre wavelets method for the solution of Fredholm integral equations. In this paper, we study the solution of Cauchy problems through Legendre wavelet based method (LWM). The Legendre wavelet based method consists of reducing the given problem to a system of simultaneous algebraic equations. The properties of Legendre wavelets are utilized to evaluate the unknown coefficients and we will find an approximate or exact solution to Eq. (1).

The organization of the paper is as follows: In section 2, we describe the basic formulation of wavelets and Legendre wavelets which are required for our subsequent development. Section 3 is devoted to the solution of Eq.(1) by using integral operator and Legendre wavelets. In section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Concluding remarks are given in the final section.

2 Properties of Legendre wavelets

2.1 Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter 'a' and the translation parameter 'b' vary continuously, we have the following family of continuous wavelets as: $\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), a, b \in R, a \neq 0$ If we restrict the parameters 'a' and 'b' to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n, k are positive integers, we have the following family of discrete wavelets: $\psi_{k,n}(t) = |a|^{-\frac{1}{2}} \psi\left((a_0)^k t - nb_0\right)$ where $\psi_{k,n}(t)$ forms an orthonormal basis. Legendre wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguements: $\hat{n} = 2n - 1, n = 1, 2, 3, ..., 2^{k-1}$, k can assume any positive integer, m is the order of Legendre polynomials and t is the normalized time. They are defined on the interval $[0, 1)$ as

$$\psi_{n,m}(t) = \left\{ \begin{array}{ll} \sqrt{m+\frac{1}{2}} 2^{\frac{k}{2}} P_m\left(2^k t - \hat{n}\right) & , for \frac{\hat{n}-1}{2^k} \leq t \leq \frac{\hat{n}+1}{2^k} \\ 0 & , otherwise \end{array} \right\} \tag{3}$$

where $m = 0, 1, 2, ..., M - 1, n = 1, 2, 3, ..., 2^{k-1}$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \hat{n}2^{-k}$. Here $P_m(t)$ are well-known Legendre polynomials of order m which are defined on the interval [-1,1], and can be determined with the aid of the following recurrence formulae: $P_0(t) = 1$, $P_1(t) = t$, $P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t\ P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t)$, where m=1,2,3,...

2.2 Function approximation

A function $u(x, t)$ defined over [0,1) may be expanded as

$$u(x,t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x) \phi_n(t) \tag{4}$$

where $\phi_n(t) = t^{n-1}$ and $c_{nm} = (2m-1) \int_0^1 \int_0^1 u(x,t) \psi_{nm}(x) \phi_n(t)\, dxdt$.
If the infinite series in Eq.(4) is truncated, then Eq.(4) can be written as

$$u(x,t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) \phi_n(t) = C^T \psi(x) \phi(t) \tag{5}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \cdots, c_{1M-1}, c_{20}, c_{21}, \cdots, c_{2M-1}, \cdots, c_{2^{K-1}0}, \cdots, c_{2^{K-1}M-1}]^T \tag{6}$$

$$\psi(x) = [\psi_{10}(x), \cdots, \psi_{1M-1}(x), \psi_{20}(x), \cdots, \psi_{2M-1}(x), \cdots, \psi_{2^{K-1}0}(x), \cdots, \psi_{2^{K-1}M-1}(x)]^T \tag{7}$$

3 Legendre Wavelet scheme for Cauchy problems

Consider the Cauchy problem given in Eq.(1). Integrating Eq. (1) with respect to 't' between 0 and t,
$\int_0^t u_t dt + \int_0^t a(x,t)u_x dt = \int_0^t f(x)dt$
$u(x,t) - u(x,0) + \int_0^t a(x,t)u_x dt = \int_0^t f(x)dt$

$$u(x,t) = u(x,0) - \int_0^t a(x,t)u_x dt + \int_0^t f(x)dt \tag{8}$$

$$Let\ u(x,t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C^T \psi(x)\phi_n(t) \tag{9}$$

where $\phi_n(t) = t^{n-1}$
Eq. (9) becomes

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C^T \psi(x)\phi_n(t) = u(x,0) - \int_0^t (\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a(x,t)(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C^T \psi(x)\phi_n(t))_x dt + \int_0^t f(x)dt \tag{10}$$

We now collocate Eq.(10) at $2^{k-1}M$ at x_i as

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C^T \psi(x_i)\phi_n(t) = u(x,0) - \int_0^t (\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a(x,t)(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C^T \psi(x_i)\phi_n(t))_x dt + \int_0^t f(x_i)dt \tag{11}$$

Eq. (11) gives $2^{k-1}M$ nonlinear equations, from which the values of C can be identified.

4 Illustrative examples

Example 1. Consider the Cauchy problem given in [5]

$$u_t(x,t) + xu_x(x,t) = 0, x \in R, t > 0 \tag{12}$$

$u(x,0) = x^2, x \in R$. We solve Eq.(12) by LWM with k=3 and M=4.

Integrating Eq. (12) with respect to 't' and applying the initial condition, we get $u(x,t) = x^2 + \int_0^t xu_x(x,t)\,dt$ Let $u(x,t) = \sum_{n=1}^{4}\sum_{m=0}^{3} c_{nm}\psi_{nm}(x)\,t^{n-1}$ and solving we get
$C_{10} = \frac{1}{3}; C_{11} = \frac{1}{2\sqrt{3}}; C_{12} = \frac{1}{6\sqrt{5}}; C_{13} = 0; C_{20} = \frac{-7}{6\sqrt{2}}; C_{21} = \frac{-3}{4\sqrt{6}}; C_{22} = \frac{-1}{12\sqrt{10}}; C_{23} = 0; C_{30} = \frac{19}{6\sqrt{2}}; C_{31} = \frac{5}{4\sqrt{6}}; C_{32} = \frac{1}{12\sqrt{10}}; C_{33} = 0; C_{40} = \frac{-37}{9\sqrt{2}}; C_{41} = \frac{-7}{6\sqrt{6}}; C_{42} = \frac{-1}{18\sqrt{10}}; C_{43} = 0;$
Hence $u(x,t) = (C_{10}\psi_{10} + \cdots + C_{13}\psi_{13}) + (C_{20}\psi_{20} + \cdots + C_{23}\psi_{23})\,t + (C_{30}\psi_{30} +$

$\cdots + C_{33}\psi_{33})\ t^2 + (C_{40}\psi_{40} + \cdots + C_{43}\psi_{43})\ t^3$ becomes
$u(x,t) = x^2 - 2x^2t + 2x^2t^2 - \frac{4}{3}x^2t^3$
For larger values of M, we get the series with infinite terms
$u(x,t) = x^2\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + ...\right)$
$u(x,t) = x^2e^{-2t}$, which is the exact solution and is in full agreement with the results given in [9].

Example 2. Consider the transport equation [1,2,5]:

$$u_t\,(x,t) + au_x\,(x,t) = 0, x \in R, t > 0 \tag{13}$$

$u\,(x,0) = x^2, x \in R$. We solve Eq.(12) by LWM with k=3 and M=4. By LWM discussed above, we get $u(x,t) = x^2 - 2atx + a^2t^2$, which is the exact solution.

Example 3. Consider the following non-homogeneous Cauchy problem [5]

$$u_t\,(x,t) + u_x\,(x,t) = x, x \in R, t > 0 \tag{14}$$

$u\,(x,0) = e^x, x \in R$. We solve Eq.(13) by LWM with k=3 and M=4. $u(x,t) = x^2 + \int_0^t xu_x\,(x,t)\,dt$
Let $u\,(x,t) = \sum_{n=1}^{4}\sum_{m=0}^{3}\ c_{nm}\psi_{nm}\,(x)\,t^{n-1}$ and solving we get
$C_{10} = \frac{41}{24}; C_{11} = \frac{7}{8\sqrt{3}}; C_{12} = \frac{3}{24\sqrt{5}}; C_{13} = \frac{1}{120\sqrt{7}}; C_{20} = \frac{-31}{24\sqrt{2}}; C_{21} = \frac{-9}{48\sqrt{6}}; C_{22} = \frac{-1}{48\sqrt{10}}; C_{23} = 0; C_{30} = \frac{5}{8\sqrt{2}}; C_{31} = \frac{1}{8\sqrt{6}}; C_{32} = 0; C_{33} = 0; C_{40} = \frac{-1}{6\sqrt{2}}; C_{41} = 0; C_{42} = 0; C_{43} = 0;$
Hence for larger value of M, we get $u\,(x,t) = (C_{10}\psi_{10} + \cdots + C_{13}\psi_{13}) + (C_{20}\psi_{20} + \cdots + C_{23}\psi_{23})\ t + (C_{30}\psi_{30} + \cdots + C_{33}\psi_{33})\ t^2 + (C_{40}\psi_{40} + \cdots + C_{43}\psi_{43})\ t^3 + ...$ becomes
$u(x,t) = t(x - \frac{t}{2}) + e^{x-t}$, which is the exact solution.

5 Conclusion

In this paper, we have applied Legendre wavelets based approximation technique for the solution of Cauchy problems. The properties of the Legendre wavelets have been used to reduce the given problem to a system of algebraic equations. Illustrative examples shows that Legendre wavelets based approximation method provides us an efficient tool to obtain the exact solutions for PDEs. Also the validity and applicability of the technique has been justified through the examples and our method is very simple and convenient to handle the partial differential equations.

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Received: July, 2012