Simply Sequentially Additive Labeling

of Some Special Trees

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Abstract

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs are becoming an increasingly useful family of Mathematical Models from a broad range of applications. Bange, Barkauskas, and Slater [1] defined a k-sequentially additive labeling f of a graph G(V,E) as a bijection from V ∪ E to \{k, k+1, …, k+|V ∪ E|−1\} such that for each edge xy ∈ E, f(xy) = f(x) + f(y). If k = 1, then G(V,E) is said to be 1-sequentially additive graph or a simply sequentially additive graph or briefly, an SSA-graph. They conjectured that all trees are 1-sequentially additive. In this paper we prove the existence of 1-sequentially additive labeling of B_{m,n}, &lt;K_{1,n} : 2&gt; and its related graph, \bigcup_{i=1}^{n} K_{i,i}, K_{1,n}(1,2,…,n), the banana trees BT(n_1, n_2), BT(n, n, n) and BT(n, n, n, n).

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1 Introduction

A labeling of a graph is assigning labels to the vertices, edges or both vertices and edges. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used. The graph labeling problem that appears in graph theory has a fast development recently. This is not only due to its mathematical importance but also because of the wide range of the applications arising from this area, for instance, x-rays, crystallography, coding theory, radar, astronomy, circuit design, and design of good Radar Type Codes, Missile Guidance Codes and Convolution Codes with optimal autocorrelation properties and communication design. Although more and more people study in this area, but there are only few general results. Most of the articles focus on particular classes of graphs or methods, so there are no useful hints and tools to solve various graph labeling problems in general. An enormous body of literature has grown around the subject in the last thirty years. They gave birth to families of graphs with attractive names such as graceful, harmonious, felicitous, elegant, cordial, magic, anti-magic and prime labeling, etc. A useful survey to know about the numerous graph labeling methods is the one by J.A. Gallian recently [2].

We consider finite undirected graphs without loops and multiple edges. For a graph $G(V,E)$, $V$ or $V(G)$ and $E$ or $E(G)$ denote the vertex set and edge set respectively.

Bange, Barkauskas, and Slater [1] defined a k-sequentially additive labeling $f$ of a graph $G(V,E)$ is a bijection from $V \cup E$ to $\{k, k+1, \ldots, k+|V \cup E|-1\}$ such that for each edge $xy$, $f(xy) = f(x) + f(y)$. If $k=1$, then $G(V,E)$ is said to be 1-sequentially additive graph or a simply sequentially additive graph or briefly, an SSA-graph. Figure 1 shows a graph with a 1-sequentially additive labeling.

![Figure 1](image)

Bange, Barkauskas, and Slater [1] and Hedge and Miller [4] proved the following results.
Theorem 1.1 [1]: (i) $K_n$ is 1-sequentially additive if and only if $n \leq 3$; (ii) $C_{3n+1}$ is not $k$-sequentially additive for $k \equiv 0$ or $2$ (mod $3$); (iii) $C_{3n+2}$ is not $k$-sequentially additive for $k \equiv 1$ or $2$ (mod $3$).

Theorem 1.2 [1]: The cycle $C_n$ is 1-sequentially additive if and only if $n \equiv 0, 1$ (mod $3$).

Theorem 1.3 [1]: The path $P_n$ is 1-sequentially additive.

Theorem 1.4 [1]: Given a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $r \leq p$, let $H$ be the graph obtained from $G$ by adding a new vertex $v_{p+1}$ of degree $r$ made adjacent to $v_1, v_2, \ldots, v_r$. If $f$ is an SSA-numbering of $G$ with $f(v_i) = i$ for $1 \leq i \leq r$, then extending $f$ by defining $f(v_{p+1}) = p+q+1$ makes $H$ simply sequentially additive.

Theorem 1.5 [4]: Let $C_{a,b}$ be a caterpillar with bipartition $\{A,B\}$ where $A = \{u_1, u_2, \ldots, u_a\}$ and $B = \{v_1, v_2, \ldots, v_b\}$, $a \leq b$. Then $C_{a,b}$ is $k$-sequentially additive for $k = a$ and $b$.

Theorem 1.6 [4]: $K_{1,n}$ is $k$-sequentially additive if and only if $k$ divides $n$.

Theorem 1.7 [4]: Denote a graph $H = G(u,X)$ with $V(H) = V(G) \cup X$ and $E(H) = E(G) \cup \{ux : x \in X\}$, where $u$ is a vertex of $G$ and $X$ is a new set of Vertices, disjoint from $V(G)$. If $G$ be a $k$-sequentially additive graph with a $k$-sequentially additive labeling $f$, then the graph $H = G(u,X)$ with $|X| = ar$, where $a = f(u)$ and $r \geq 1$, is also a $k$-sequentially additive graph.

Bange, Barkauskas, and Slater [1] conjectured that all trees are 1-sequentially additive.

In this paper we prove that the trees $B_{m,n}$, $<K_{1,n} : 2>$ and related graphs, $\bigcup_{i=1}^{n} K_{1,i}$, $K_{1,n}(1,2,\ldots,n)$, the banana trees $BT(n_1, n_2)$, $BT(n, n, n)$ and $BT(n, n, n, n)$ admit 1-sequentially additive labeling. The reader is directed to Harary [3] for all additional notation and terminology not provided in this paper.

2 Main Results

Now we provide the definitions for the graphs to be discussed in this paper. $B_{n,n}$ is the $n$-bistar obtained from two disjoint copies of $K_{1,n}$ by joining the center vertices by an edge. $B_{m,n}$ is the bistar obtained from two disjoint copies of $K_{1,m}$ and $K_{1,n}$ by joining the center vertices by an edge. The tree $<K_{1,n} : 2>$ is obtained from the bistar $B_{n,n}$ by subdividing the edge joining the two stars. The graph $B(r, s, t)$ is obtained from a path of length $t$ by attaching the stars $K_{1,r}$ and $K_{1,s}$ with its pendant vertices.

Next we define $K_{1,n}(1,2,\ldots,n)$ is a graph obtained from $K_{1,n}$ of center vertex $v_0$ and end vertices $v_1, v_2, \ldots, v_n$ by joining $i$ pendant vertices at each $v_i$, $i=1, 2, \ldots, n$.

Throughout the paper we define the bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)|+|E(G)|\}$. We denote the greatest integer less than or equal to the real number $x$ by $[x]$. 

Theorem 2.1: The bistar $B_{m,n}$ is 1-sequentially additive.

Proof: \(G(V,E) = B_{m,n}\). Then \(G\) has \((m+n+2)\) vertices and \((m+n+1)\) edges. We define the bijection \(f: V \cup E \rightarrow \{1, 2, \ldots, 2(m+n)+3\}\) as follows:

**Case 1:** \(m\) or \(n\) or both \(m\) and \(n\) are even.

Let \(n\) be even.

Let \(V = V_1 \cup V_2 \cup V_3\), where \(V_1 = \{v_1, v_2\}\), \(V_2 = \{v_{1,i} ; 1 \leq i \leq m\}\), \(V_3 = \{v_{2,j} ; 1 \leq j \leq n\}\) with degree of \(v_1\) is \((m+1)\) and degree of \(v_2\) is \((n+1)\) and \(E = E_1 \cup E_2 \cup E_3\), where \(E_1 = \{v_1v_2\}\), \(E_2 = \{v_{1,i}v_{1,j} ; 1 \leq i \leq m\}\), \(E_3 = \{v_{2,j}v_{2,j} ; 1 \leq j \leq n\}\).

Define

\[
\begin{align*}
    f(v_1) &= 1, \\
    f(v_2) &= 2, \\
    f(v_{1,i}) &= 2(i+1) ; 1 \leq i \leq m \\
    f(v_{2,j}) &= \begin{cases} 
    2(m + 2j) ; & 1 \leq j \leq \frac{m}{2} \\
    2(m-n+2j)+1 ; & \frac{m}{2} < j \leq n 
    \end{cases} \\
    f(v_1v_2) &= 3 \\
    f(v_{1,i}v_{1,j}) &= 2i+3 ; 1 \leq i \leq m \\
    f(v_{2,j}v_{2,j}) &= \begin{cases} 
    2(m+2j+1) ; & 1 \leq j \leq \frac{n}{2} \\
    2(m-n+2j)+3 ; & \frac{n}{2} < j \leq n. 
    \end{cases}
\end{align*}
\]

The labels are distinct and satisfy the condition \(f(uv) = f(u) + f(v)\) for each \(uv \in E\). Hence \(B_{m,n}\) has 1-sequentially additive labeling, when \(m\) or \(n\) or both \(m, n\) are even.

**Case 2:** Both \(m, n\) are odd.

**Subcase 2.1:** \(n \equiv 0 \pmod{3}\).

Let \(V\) and \(E\) be defined as in case 1.

Define

\[
\begin{align*}
    f(v_1) &= 1, \\
    f(v_2) &= m+n+1, \\
    f(v_{1,i}) &= \begin{cases} 
    n + 2i ; & 1 \leq i \leq \frac{m-1}{2} \\
    2(n + i + 1) ; & \frac{m-1}{2} < i \leq m 
    \end{cases} \\
    f(v_{2,j}) &= j+1 ; 1 \leq j \leq n. \\
    f(v_1v_2) &= 3 \\
    f(v_{1,i}v_{1,j}) &= \begin{cases} 
    n+1+2i ; & 1 \leq i \leq \frac{m-1}{2} \\
    2(n+i)+3 ; & \frac{m-1}{2} < i \leq m 
    \end{cases} \\
    f(v_{2,j}v_{2,j}) &= (m+n+2) + j ; 1 \leq j \leq n.
\end{align*}
\]

The labels are distinct and satisfy the conditions \(f(uv) = f(u) + f(v)\) for each edge \(uv \in E\). Hence the bistar \(B_{m,n}\) admits 1-sequentially additive labeling, when \(n\) is odd and \(n \equiv 0 \pmod{3}\). In Fig. 2 we give the 1-sequentially additive labeling for the bistar \(B_{3,3}\).
**Subcase 2.2: n ≡ 1 (mod 3).**

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \), where

\( V_1 = \{v_1, v_2\}, \)

\( V_2 = \{v_{1,i} ; 1 \leq i \leq m\}, \)

\( V_3 = \{u_{k,j} ; 1 \leq k \leq \frac{n-1}{3} \text{ and } 1 \leq j \leq 3\} \)

\( V_4 = \{u_{k,j} ; k = \frac{n+2}{3}\} \) and

\( E = E_1 \cup E_2 \cup E_3 \cup E_4 \), where

\( E_1 = \{v_1v_2\}, \)

\( E_2 = \{v_1v_{1,i} ; 1 \leq i \leq m\}, \)

\( E_3 = \{v_2u_{k,j} ; 1 \leq k \leq \frac{n-1}{3}, 1 \leq j \leq 3\} \) and

\( E_4 = \{v_2u_{k,j} ; k = \frac{n+2}{3}\}. \)

Define

\[ f(v_1) = 1, \quad f(v_2) = 3, \quad f(v_{1,i}) = 2(n+i+1); 1 \leq i \leq m, \]

\[ f(u_{k,j}) = \begin{cases} 6k + j - 1; & 1 \leq k \leq \frac{n-1}{3} \text{ and } 1 \leq j \leq 3 \\ 2; & k = \frac{n+2}{3} \text{ and } j = 1 \end{cases} \]

\[ f(v_1v_2) = 4, \quad f(v_1v_{1,i}) = 2(n+i)+3; 1 \leq i \leq m, \]

\[ f(v_2u_{k,j}) = \begin{cases} 6k + j + 2; & 1 \leq k \leq \frac{n-1}{3} \text{ and } 1 \leq j \leq 3 \\ 5; & k = \frac{n+2}{3} \text{ and } j = 1 \end{cases} \]

The labels are distinct and satisfy the conditions \( f(uv) = f(u) + f(v) \) for each edge \( uv \in E \). Hence \( B_{m,n} \) admits 1-sequentially additive labeling, when \( m, n \) are odd and \( n \equiv 1 (\text{mod } 3) \).

**Subcase 2.3: n ≡ 2 (mod 3).**

Let \( V \) and \( E \) be defined as in case 1.

Define

\[ f(v_1) = m+n+1, \quad f(v_2) = 1, \quad f(v_{1,i}) = i+1; 1 \leq i \leq m, \]

\[ f(v_{2,j}) = \begin{cases} m + 2j; & 1 \leq j \leq \frac{n}{2} \\ 2(m + j + 1); & \frac{n}{2} \leq j \leq n \end{cases} \]

\[ f(v_1v_2) = m+n+2, \quad f(v_1v_{1,i}) = (m+n+2) + i; 1 \leq i \leq m, \]

\[ f(v_{2,j}) = \begin{cases} m + 2j + 1; & 1 \leq j \leq \frac{n}{2} \\ 2(m + j + 3); & \frac{n}{2} \leq j \leq n \end{cases} \]

The labels are distinct and satisfy the conditions \( f(uv) = f(u) + f(v) \) for each edge \( uv \in E \). Hence the bistar \( B_{m,n} \) admits 1-sequentially additive labeling \( m, n \) are odd and \( n \equiv 2 (\text{mod } 3) \).

In all the cases \( B_{m,n} \) has a 1-sequentially additive labeling. Hence \( B_{m,n} \) is 1-sequentially additive.

In Fig. 3, we give the 1-sequentially additive labeling for the bistar \( B_{7,11} \).

![Fig. 3: B_{7,11}](image-url)
In the following theorem we prove \( <K_{1,n} : 2> \) is a 1-sequentially additive graph.

**Theorem 2.2:** The tree \( <K_{1,n} : 2> \) is 1-sequentially additive.

**Proof:** Let \( V \) be the vertex set and \( E \) be the edge set of \( <K_{1,n} : 2> \). Then \(|V| = 2n+3, |E| = 2n+2\). We define bijection \( f : V \cup E \rightarrow \{1, 2, \ldots, 4n+5\} \) as follows.

**Case 1:** \( n \) is even.

Let \( V = V_1 \cup V_2 \cup V_3 \), where \( V_1 = \{v_1, v_2, u_0\}, V_2 = \{v_{1,i} ; 1 \leq i \leq n\}, V_3 = \{v_{2,j} ; 1 \leq j \leq n\} \), \( \deg(v_1) = \deg(v_2) = n+1 \) and \( \deg(u_0) = 2 \) and all the other vertices are of degree one and \( E = E_1 \cup E_2 \cup E_3 \), where \( E_1 = \{v_1u_0, v_2u_0\}, E_2 = \{v_1v_{1,i} ; 1 \leq i \leq n\}, E_3 = \{v_2v_{2,j} ; 1 \leq j \leq n\} \).

Define

\[
f(v_1) = 1, \quad f(v_2) = 2, \quad f(u_0) = 3, \quad f(v_{1,i}) = 2(i+2) ; 1 \leq i \leq n, \\
f(v_1u_0) = 4, \quad f(v_2u_0) = 5, \\
f(v_{1,i}v_{1,j}) = \begin{cases} 
2(2n+2j+1); & 1 \leq j \leq \frac{n}{2} \\
3+4j; & \left(\frac{n}{2}+1\right) \leq j \leq n 
\end{cases}
\]

\[
f(v_2v_{2,j}) = \begin{cases} 
2(n+2j+2); & 1 \leq j \leq \frac{n}{2} \\
4j+5; & \left(\frac{n}{2}+1\right) \leq j \leq n 
\end{cases}
\]

**Case 2:** \( n \) is odd.

**Subcase 2.1:** \( n \equiv 0 \pmod{3} \)

Let \( V = V_1 \cup V_2 \cup V_3 \cup V_4 \), where \( V_1 = \{v_1, v_2, u_0\}, V_2 = \{v_{1,i} ; 1 \leq i \leq n\}, V_3 = \{u_{i,j} ; 1 \leq i \leq \left\lfloor \frac{n}{6} \right\rfloor \text{ \& } 1 \leq j \leq 6\}, V_4 = \{u_{i,j} ; i = \left\lceil \frac{n}{6} \right\rceil + 1 \text{ \& } 1 \leq j \leq 3\} \) and \( E = E_1 \cup E_2 \cup E_3 \cup E_4 \) where \( E_1 = \{v_1u_0, v_2u_0\}, E_2 = \{v_1v_{1,i} ; 1 \leq i \leq n\}, E_3 = \{v_2u_{i,j} ; 1 \leq i \leq \left\lceil \frac{n}{6} \right\rceil \text{ \& } 1 \leq j \leq 6\} \) and \( E_4 = \{v_2u_{i,j} ; i = \left\lceil \frac{n}{6} \right\rceil + 1 \text{ \& } 1 \leq j \leq 3\} \).

Define

\[
f(v_1) = 1, \quad f(u_0) = 4, \quad f(v_2) = 6, \\
f(v_{1,i}) = \begin{cases} 
11; & i = 1 \\
2(n+i+2); & 3 \leq i \leq n 
\end{cases}
\]

\[
f(u_{i,j}) = \begin{cases} 
12i+j+3; & 1 \leq i \leq \left\lfloor \frac{n}{6} \right\rfloor \text{ \& } 1 \leq j \leq 6 \\
j+6; & i = \left\lceil \frac{n}{6} \right\rceil + 1 \text{ \& } 1 \leq j \leq 3 
\end{cases}
\]

\[
f(v_1u_0) = 5, \quad f(v_2u_0) = 10, \\
f(v_1v_{1,i}) = \begin{cases} 
3; & i = 1 \\
12; & i = 2 \\
2(n+i+5); & 3 \leq i \leq n 
\end{cases}
\]

\[
f(v_2v_{2,j}) = 12i+j+9; 1 \leq i \leq \left\lfloor \frac{n}{6} \right\rfloor \text{ \& } 1 \leq j \leq 6 \\
f(v_2u_{i,j}) = j+12; \quad i = \left\lceil \frac{n}{6} \right\rceil + 1 \text{ \& } 1 \leq j \leq 3.
\]

**Subcase 2.2:** (i) \( n \equiv 1 \pmod{3} \) and \( n \neq 1 \).

Let \( V \) and \( E \) be defined as in case 1.
We define \( f \) on \( V \cup E \) as follows:

\[
\begin{align*}
\text{f(v1)} &= 1, \quad \text{f(u0)} = n+3, \quad \text{f(v2)} = n+5, \\
2i; & \quad 1 \leq i \leq \frac{n+1}{2} \\
n + 6; & \quad i = \frac{n+3}{2} \\
f(v_{1,j}) &= 2n + 9; \quad i = \frac{n+5}{2} \\
2n + 11; & \quad i = \frac{n+7}{2} \\
2(n + i + 2); & \quad \frac{n+9}{2} \leq i \leq n
\end{align*}
\]

\[
\begin{align*}
f(v_{1u0}) &= n + 4, \quad \text{f(v2u0)} = 2n + 8, \quad \text{f(v1v_{1,i})} = \\
2i + 1; & \quad 1 \leq i \leq \frac{n+1}{2} \\
n + 7; & \quad i = \frac{n+3}{2} \\
2n + 10; & \quad i = \frac{n+5}{2} \\
2n + 12; & \quad i = \frac{n+7}{2} \\
2(n + i) + 5; & \quad \frac{n+9}{2} \leq i \leq n
\end{align*}
\]

\[
\begin{align*}
f(v_{2,j}) &= 2n + 12 + j; \quad 1 \leq j \leq n.
\end{align*}
\]

(ii). If \( n = 1 \), then \(<K_{1,n} : 2>\) is a Path \( P_5 \). By Theorem 1.3[1], it is an SSA-graph.

**Subcase 2.3:** \( n \equiv 2 \text{ (mod 3)} \).

Let \( V \) and \( E \) be defined as in case 1.

Define \( f \) on \( V \cup E \) as follows:

\[
\begin{align*}
\text{f(v1)} &= 1, \quad \text{f(v2)} = n+5, \quad \text{f(u0)} = n+3, \quad \text{f(v_{1,i})} = \\
2i; & \quad 1 \leq i \leq \frac{n+1}{2} \\
2(n + 3); & \quad i = \frac{n+3}{2} \\
2n + 9; & \quad i = \frac{n+5}{2} \\
2n + 12; & \quad i = \frac{n+7}{2} \\
2(n + 2 + i); & \quad \frac{n+7}{2} \leq i \leq n
\end{align*}
\]

\[
\begin{align*}
f(v_{2,j}) &= n+5 + j; \quad 1 \leq j \leq n. \\
f(v_{1u0}) &= n + 4, \quad \text{f(v2u0)} = 2n + 8 \\
2i + 1; & \quad 1 \leq i \leq \frac{n+1}{2} \\
2n + 7; & \quad i = \frac{n+3}{2} \\
2n + 10; & \quad i = \frac{n+5}{2} \\
2(n + i) + 5; & \quad \frac{n+7}{2} \leq i \leq n
\end{align*}
\]

\[
\begin{align*}
f(v_{2v2,j}) &= 2n + 10 + j; \quad 1 \leq j \leq n.
\end{align*}
\]

In all the above cases, the labels are distinct and satisfy the condition \( f(uv) = f(u) + f(v) \) for each \( uv \in E \). Hence \(<K_{1,n} : 2>\) admits 1-sequentially additive labeling.

In Fig. 4, we display 1-sequentially additive labeling for \(<K_{1,5} : 2>\).
Remark 2.1: It follows from the Theorem 1.5[4], \(<K_{1,n} : 2>\) is a 2- sequentially additive and as well as \((2n+1)\) - sequentially additive graph.

Theorem 2.3: Let \(G(V,E)\) be the graph obtained from the tree \(<K_{1,n} : 2>\) by appending two pendent edges to a new vertex which subdivides the edge joining the centre vertices of the two copies of \(k_{1,n}\) in \(<K_{1,n} : 2>\). Then \(G\) is an SSA graph.

Proof: The graph \(G\) has \(2n+5\) vertices and \(2n+4\) edges. We define a map \(f : V \cup E \rightarrow \{1, 2, \ldots, 4n+9\}\) as follows:

**Case 1:** \(n\) is even.

Let the vertex set \(V(G) = V_1 \cup V_2 \cup V_3\), where \(V_1 = \{w_i : 0 \leq i \leq 4\}\), \(V_2 = \{u_i : 1 \leq i \leq n\}\), \(V_3 = \{v_i : 1 \leq i \leq n\}\) and the edge set \(E(G) = E_1 \cup E_2 \cup E_3\), where \(E_1 = \{w_0w_i : 1 \leq i \leq 4\}\), \(E_2 = \{w_1u_i : 1 \leq i \leq n\}\), \(E_3 = \{w_2v_i : 1 \leq i \leq n\}\).

Define

\[
\begin{align*}
    f(w_0) &= 5, \\
    f(w_i) &= i, 1 \leq i \leq 4, \\
    f(u_i) &= 2(i+4), 1 \leq i \leq n, \\
    f(v_i) &= \begin{cases} 2(n + 2i + 3) & 1 \leq i \leq \frac{n}{2} \\ 4i + 7 & \left(\frac{n}{2}+1\right) \leq i \leq n \end{cases} \\
    f(w_0w_i) &= 5+i, 1 \leq i \leq 4, \\
    f(w_1u_i) &= 2i+9, 1 \leq i \leq n, \\
    f(w_2v_i) &= \begin{cases} 2(n + 2i + 4) & 1 \leq i \leq \frac{n}{2} \\ 4i + 9 & \left(\frac{n}{2}+1\right) \leq i \leq n \end{cases}
\end{align*}
\]

**Case 2:** \(n\) is odd.

**Subcase 2.1:** \(n \equiv 0 \pmod{3}\)

Let \(V(G) = V_1 \cup V_2 \cup V_3\) and \(E(G) = E_1 \cup E_2 \cup E_3\), where \(V_1, V_2, E_1\) and \(E_2\) are as defined in case 1 and \(V_3 = \{v_{k,j} : 1 \leq k \leq \frac{n}{3}, 1 \leq j \leq 3\}\),

\(E_3 = \{w_2v_{k,j} : 1 \leq k \leq \frac{n}{3}, 1 \leq j \leq 3\}\).

Define

\[
\begin{align*}
    f(w_1) &= 1, f(w_2) = 3, f(w_3) = 2, f(w_4) = 4, f(w_0) = 5, \\
    f(u_i) &= 2(i+4), 1 \leq i \leq n, \\
    f(v_{k,j}) &= 2n+6k+j+3, 1 \leq k \leq \frac{n}{3}, 1 \leq j \leq 3, \\
    f(w_0w_1) &= 6, f(w_0w_2) = 8, f(w_0w_3) = 7, f(w_0w_4) = 9
\end{align*}
\]
Simply sequentially additive labeling

Let $f(w_1u_i) = 2i + 9$; $1 \leq i \leq n$

$f(w_2v_{k,j}) = 2n + 6k + j + 6$; $1 \leq k \leq \frac{n}{3}$, $1 \leq j \leq 3$

**Subcase 2.2: $n \equiv 1 \pmod{3}$**

Let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{w_i; 0 \leq i \leq n\}$, $V_2 = \{u_i; 1 \leq i \leq n\}$, $V_3 = \{v_{k,j}; 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor, 1 \leq j \leq 3\}$, $V_4 = \{v_{k,1}; k = \frac{2n}{3}\}$

and $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$, where

$E_1 = \{w_0w_i; 1 \leq i \leq 4\}$, $E_2 = \{w_1u_i; 1 \leq i \leq n\}$, $E_3 = \{w_2v_{k,j}; 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor, 1 \leq j \leq 3\}$ and $E_4 = \{w_2v_{k,1}; k = \frac{2n}{3}\}$

For $0 \leq i \leq 4$, assign the label to $w_i$ and $w_0w_i$ as in subcase 2.1.

**Subcase 2.3: $n \equiv 2 \pmod{3}$**

Let $V(G)$ and $E(G)$ be defined as in case 1.

$f(w_0) = n + 3$, $f(w_1) = 1$, $f(w_2) = n + 5$, $f(w_3) = 4$, $f(w_4) = 5$

$\begin{cases} 2; & i = 1 \\ 2(i + 1); & 2 \leq i \leq \frac{n-1}{2} \\ 2n + 9; & i = \frac{n-1}{2} \\ 2n + 12; & i = \frac{n+1}{2} \\ (5n + 2i + 21)/2; & \frac{n+5}{2} \leq i \leq n \end{cases}$

For $0 \leq i \leq 4$

$f(w_0u_i) = \begin{cases} 12; & i = 1 \\ 2i + 11; & 2 \leq i \leq n \end{cases}$

$f(w_2v_{k,j}) = \begin{cases} 2n + 6k + j + 8; & 1 \leq k \leq \frac{n-1}{3}; 1 \leq j \leq 3 \\ 13; & k = \frac{2n}{3}; j = 1 \end{cases}$

$f(w_0w_1) = n + 4$, $f(w_0w_2) = 2n + 8$

$f(w_0w_3) = n + 7$, $f(w_0w_4) = n + 8$

$\begin{cases} 3; & i = 1 \\ 2i + 3; & 2 \leq i \leq \frac{n-1}{2} \\ 2n + 10; & i = \frac{n-1}{2} \\ 2n + 13; & i = \frac{n+1}{2} \\ (5n + 2i + 23)/2; & \frac{n+5}{2} \leq i \leq n \end{cases}$

$f(w_2v_{i,j}) = \begin{cases} 2n + 11; & i = 1 \\ 2n + i + 12; & 2 \leq i \leq n \end{cases}$
In all the above cases, the labels are distinct and satisfy the condition \( f(uv) = f(u) + f(v) \) for each \( uv \in E \). Hence \( G \) admits an 1-sequentially additive labeling. SSA-labeling of the graph \( G \) in Theorem 2.3 with \( n = 6 \) is given in Fig. 5.

We denote by \( \bigcup_{i=1}^{n} K_{1,i} \) the union of \( n \) disjoint copies of \( K_{1,i} \) for \( i = 1, 2, \ldots, n \).

**Theorem 2.4:** The graph \( \bigcup_{i=1}^{n} K_{1,i} \) is an SSA graph.

**Proof.** Let \( V \) be the vertex set and \( E \) be the edge set of \( \bigcup_{i=1}^{n} K_{1,i} \). Then \( |V| + |E| = n^2 + 2n \). \( V \) and \( E \) be defined as \( V = \{v_{ij} ; 1 \leq i \leq n \text{ and } 1 \leq j \leq i\} \cup \{v_i ; 1 \leq i \leq n\} \), \( E = \{v_iv_{ij} ; 1 \leq i \leq n \text{ and } 1 \leq j \leq i\} \). We define the bijection \( f : V \cup E \rightarrow \{1, 2, \ldots, n^2 + 2n\} \) as follows.

\[
\begin{align*}
f(v_i) &= i; 1 \leq i \leq n, \\
f(v_{ij}) &= n + i^2 - i + j \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq i, \\
f(v_iv_{ij}) &= n + i^2 + j \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq i.
\end{align*}
\]

Therefore the labels are distinct and satisfy the condition \( f(uv) = f(u) + f(v) \) for each edge \( uv \in E \). Hence \( \bigcup_{i=1}^{n} K_{1,i} \) is an SSA-graph.

The graph \( K_{1,n}(1,2,\ldots,n) \) which we defined earlier is the same as the graph obtained from \( \bigcup_{i=1}^{n} K_{1,i} \) by joining the centre vertex \( v_i \) of each \( K_{1,i} \) \( (i = 1, 2, \ldots, n) \) by an edge to a new vertex \( v_0 \).

By Theorem 1.4 [1] and Theorem 2.4, we have the following the corollary.

**Corollary 2.5:** The graph \( K_{1,n}(1,2,\ldots,n) \) is an SSA-graph.

**Proof:** The labeling \( f \) defined on \( \bigcup_{i=1}^{n} K_{1,i} \) in theorem 3.2 is extended to the vertex \( v_0 \)

and the edges adjacent to \( v_0 \) by \( f(v_0) = (n+1)^2 \) and \( f(v_0v_i) = (n+1)^2 + i \) for \( 1 \leq i \leq n \), we have \( K_{1,n}(1,2,\ldots,n) \) is an SSA-graph.
Simply sequentially additive labeling

SSA-labeling of the graph $K_{1,n}(1,2,\ldots,n)$ in Corollary 2.5 with $n=5$ is shown in Figure 6.

In the following theorems, we present an SSA-labeling of some particular types of banana trees. The banana tree is defined as follows.

Let $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_k}$ be a family of disjoint stars with the vertex sets $V(K_{1,n_i}) = \{c_i, v_{i,1}, \ldots, v_{i,n_i}\}$ and $\deg(c_i) = n_i$, $1 \leq i \leq k$. A banana tree $BT(n_1, n_2, \ldots, n_k)$ is a tree obtained by adding a new vertex and joining it to $v_{1,1}, v_{2,1}, \ldots, v_{k,1}$.

Denote the vertex and edge sets of $G(V,E) \cong BT(n_1, n_2, \ldots, n_k)$ as follows:

$V(G) = \{v\} \cup \{c_i; 1 \leq i \leq k\} \cup \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n_i\}$

$E(G) = \{vv_{i,1}; 1 \leq i \leq k\} \cup \{civ_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n_i\}$.

**Theorem 2.6:** The banana tree $BT(n_1, n_2)$ admits an SSA-labeling.

**Proof:** Let $G = BT(n_1, n_2)$. We consider two cases.

**Case 1:** $n_1 \geq n_2 > 1$

$V(G) = \{v, c_1, c_2\} \cup \{v_{i,j}; 1 \leq i \leq 2, 1 \leq j \leq n_i\}$

$E(G) = \{v_{v_i,1}; 1 \leq i \leq 2\} \cup \{c_i v_{i,j}; 1 \leq i \leq 2, 1 \leq j \leq n_i\}$

Then $|V(G)| = n_1 + n_2 + 3$ and $|E(G)| = n_1 + n_2 + 2$.

Define a labeling $f: V \cup E \to \{1, 2, \ldots, 2(n_1+n_2) + 5\}$ as follows:

For $1 \leq i \leq 2$

$f(v) = 2$, $f(c_i) = 2i-1$

$f(v_{v_i,1}) = 3i+1$

$f(v_{v_i,2}) = 4(4-i)$

$f(v_{i,j}) = 4j-i+4$, $3 \leq j \leq n_2$

$f(v_{i,j}) = 2(n_2+j+2)$; $n_2+1 \leq j \leq n_1$

$f(vv_{i,1}) = 3(i+1)$

$f(c_i v_{i,j}) = \begin{cases} 5i; & j = 1 \\ 15-2i; & j = 2 \\ i+4j+3; & 3 \leq j \leq n_2. \end{cases}$

$f(c_i v_{i,j}) = 2n_2 + 2j + 5$; $n_2 + 1 \leq j \leq n_1$.

The labels defined above are all distinct and satisfy the condition $f(uv) = f(u) + f(v)$ for each $uv \in E(G)$ and for $n_1 \geq n_2 > 1$. 
Case 2. \( n_1 \geq n_2 = 1 \). Then \( V(G) = \{v\} \cup \{c_1, c_2\} \cup \{v_{1,j} \mid 1 \leq j \leq n_1\} \cup \{v_{2,1}\} \) and 
\( E(G) = \{vv_{1,1}, vv_{2,1}\} \cup \{c_2v_{2,1}\} \cup \{c_1v_{1,j} \mid 1 \leq j \leq n_1\} \)
and \(|V(G) \cup E(G)| = 2n_1 + 7\).

We define the labeling \( f : V(G) \cup E(G) \rightarrow \{1, 2, ... , 2n_1 + 7\} \) as follows:
\[ f(c_i) = i, i = 1, 2; \]
\[ f(v) = 3, \]
\[ f(v_{1,1}) = 2(i+1); i = 1, 2; \]
\[ f(v_{1,j}) = 6+2j; 2 \leq j \leq n_1, \]
\[ f(v_{2,1}) = 2i+5; i = 1, 2 \]
\[ f(c_{1v_{1,j}}) = 3i+2; i = 1, 2, \]
\[ f(c_{1v_{1,j}}) = 2j+7; 2 \leq j \leq n_1. \]

In both cases, the labels defined on \( V(G) \cup E(G) \) are all distinct and satisfy the condition \( f(uv) = f(u) + f(v) \) for each \( uv \in E(G) \). Hence the graph \( BT(n_1, n_2) \) admits an SSA-labeling.

An SSA-labeling of \( BT(6, 6) \) is shown in Figure 7.

\[ \text{Fig. 7: SSA – labeling of } BT(6,6) \]
Theorem 2.8: The banana tree BT(n, n, n, n) is an SSA-graph.

Proof: Let G(V,E) = BT(n, n, n, n) and V(G) = \{v\} \cup \{c_i / 1 \leq i \leq 4\} \cup \{v_{ij} / 1 \leq i \leq 3, 1 \leq j \leq n\} and E(G) = \{vv_{ij}\} \cup \{c_{ij}v_{ij} / 1 \leq i \leq 4, 1 \leq j \leq n\} respectively. Then |V(G) \cup E(G)| = 8n+9. Then |V(G) \cup E(G)| = 8n+9. We define a bijection f : V(G) \cup E(G) \rightarrow \{1, 2, ..., 8n+9\} as follows:

Case 1: n \geq 2.
For the vertices
f(v) = 1, f(c_i) = i+1; 1 \leq i \leq 4.
f(v_{1,1}) = 17, f(v_{2,1}) = 13, f(v_{3,1}) = 8, f(v_{4,1}) = 6
f(v_{1,2}) = 21, f(v_{2,2}) = 22, f(v_{3,2}) = 20, f(v_{4,2}) = 10
f(v_{1,j}) = 8j+6; 3 \leq j \leq n.
f(v_{ij}) = i+8j; 2 \leq i \leq 4, 3 \leq j \leq n.
f(vv_{1,1}) = 18, f(vv_{2,1}) = 14, f(vv_{3,1}) = 9, f(vv_{4,1}) = 7
f(c_{1v_{1,1}}) = 19, f(c_{2v_{2,1}}) = 16, f(c_{3v_{3,1}}) = 12, f(c_{4v_{4,1}}) = 11
f(c_{1v_{1,2}}) = 23, f(c_{2v_{2,2}}) = 25, f(c_{3v_{3,2}}) = 24, f(c_{4v_{4,2}}) = 15
f(c_{ijv_{ij}}) = 8(j+1); 3 \leq j \leq n
f(c_{ijv_{ij}}) = 2i+8j+1; 2 \leq i \leq 4, 3 \leq j \leq n
Clearly the labels defined on V \cup E satisfy the condition f(uv) = f(u) + f(v) for each uv \in E(G).

Case 2: Let n = 1. An SSA-labeling of BT(1,1,1,1) is shown in Figure 9.

Thus in both cases BT(n, n, n, n) admits an SSA-labeling.
For example, SSA-labeling of B(6, 6, 6, 6) is shown in Fig. 10.

Fig. 10: SSA-labeling of BT(6,6,6,6)

References


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