Total Co-Independent Domination in Graphs

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Abstract

A set $D$ of vertices in a graph $G$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A dominating set $D$ of a graph $G$ is total dominating set if the induced subgraph $\langle D \rangle$ has no isolated vertices. In this paper, we introduce the total co-independent domination in graphs, exact value for some standard graphs, bounds and some results are established.

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1 Introduction

All graphs in this paper will be finite and undirected, without loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph $G$, respectively. In general, we use $\langle X \rangle$ to denote the subgraph
induced by the set of vertices $X$. $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex $v$, respectively. A set $D$ of vertices in a graph $G$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. If $G$ is connected graph, then a vertex cut of $G$ is a subset $R$ of $V$ with the property that the subgraph of $G$ induced by $V - R$ is disconnected. If $G$ is not a complete Graph, then the vertex connectivity number $k(G)$ is the minimum cardinality of a vertex cut. If $G$ is complete graph $K_p$ it is known that $k(G) = p - 1$.

For terminology and notations not specifically defined here we refer reader to [4]. For more details about domination number and its related parameters, we refer to [5], [11], and [13].

A dominating set $S$ of $G$ is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set of $G$ is called the connected domination number of $G$ and is denoted by $\gamma_c(G)$ [12].

A dominating set $S$ of $G$ is called non-split dominating set if the induced subgraph $\langle V - S \rangle$ is connected. The minimum cardinality of a non-split dominating set of $G$ is called the non split domination number of $G$ and is denoted by $\gamma_{ns}(G)$ [8].

A dominating set $S$ of $G$ is called total dominating set if the induced subgraph $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of $G$ is called the total domination number of $G$ and is denoted by $\gamma_t(G)$ [5].

Many application of dominations in graphs can be extended to the total co-independent domination. For example the routing protocols in such networks are typically based on the concept of a virtual backbone, which is a (small) subset of nodes that are used as a core for communication within the network. In particular, totally connected dominating sets are often used to describe a virtual backbone in ad hoc wireless networks. This motivates us to introduce the concept of total co-independent domination in a graph.

2 Total Co-independent Domination Number

**Definition.** A total dominating set $S$ of a graph $G = (V, E)$ is called total co-independent dominating set if the induced subgraph $\langle V - S \rangle$ has no edge and has at least one vertex. The minimum cardinality of a total co-independent dominating set of $G$ is called the total co-independent domination number of $G$ and is denoted by $\gamma_{t,coi}(G)$. A total co-independent dominating set $S$ is said
to be minimal if no proper subset of $S$ is total co-independent dominating set.

**Example 2.1** Let $G$ be the graph in the Figure 1, $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

![Figure 1: G](image)

\(\{2, 3\}\) is the minimum total dominating set. Hence $\gamma_t(G) = 2$, clearly $\{2, 3\}$ is not total co-independent set of $G$. The minimum total co-independent dominating sets are $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{2, 3, 4, 9\}$, $\{2, 3, 5, 9\}$. Therefore $\gamma_{t,\text{coi}}(G) = 4$.

As in the standard dominating set, any minimum total co-independent dominating set is minimal, but the converse is not true as seen in Figure 1. The set $\{1, 2, 4, 5, 6, 7\}$ is minimal total co-independent dominating set but not minimum total co-independent dominating set.

**Observation 2.2** A non empty graph $G$ is without isolated vertices if and only if it admits a total co-independent dominating set.

**Theorem 2.3** A total co-independent dominating set $D$ of a graph $G$ is minimal if and only if for each vertex $v \in D$, one of the following conditions is satisfied: (i) There exist a vertex $u \in V$ such that $N(u) \cap D = \{v\}$.

(ii) For every vertex $v$ there exist $w \in V - D$ adjacent to $v$.

**Proof.** Let $D$ be a total co-independent dominating set of $G$. Suppose there exist a vertex $v \in D$ does not satisfy any of the conditions. Then Clearly by (i) $D - \{v\}$ is total dominating set and also by (ii) $V - (D - \{v\})$ is independent set. Therefore $D - \{v\}$ is total co-independent dominating set of $G$, a contradiction. Hence one of the given conditions is satisfied. The converse is straightforward.
The following observations are immediate.

**Observation 2.4** For any Cycle $C_p$, $\gamma_{t,coi}(C_p) = p - \left\lfloor \frac{p}{3} \right\rfloor$.

**Observation 2.5** For any path $P_p$, $\gamma_{t,coi}(P_p) = p - \left\lfloor \frac{p}{3} \right\rfloor$.

**Observation 2.6** For any wheel $W_p$ with $p$ vertices, $\gamma_{t,coi}(W_p) = 1 + \left\lceil \frac{p-1}{2} \right\rceil$.

**Observation 2.7** For any complete Graph $K_p$, $\gamma_{t,coi}(K_p) = p - 1$.

**Observation 2.8** For any complete bipartite graph $K_{r,s}$ where $r \leq s$, $\gamma_{t,coi}(K_{r,s}) = r + 1$.

**Proposition 2.9** Let $G$ be graph with $p \geq 3$ vertices and has no isolated vertices, then

$$2 \leq \gamma_{t,coi}(G) \leq p - 1.$$ 

Further the equality of upper bound is attained if $G$ is $P_3$ or $G$ is two star, and the upper bound attained if $G$ is complete graph $K_p$ or $G = P_2 \cup P_3$.

**Proof.** Let $G$ be graph with $p \geq 3$ vertices and has no isolated vertices and $D$ total co-independent dominating set of $G$. Then obviously $D$ is total total dominating set. Hence $2 \leq \gamma_{t,coi}(G)$. For the upper bound, suppose that $D = V - \{u\}$ where $u$ is a pendent vertex with respect to some spanning tree of $G$. Clearly $D$ is total co-independent dominating set of $G$. Therefore $\gamma_{t,coi}(G) \leq p - 1$.

Hence $2 \leq \gamma_{t,coi}(G) \leq p - 1$.

**Proposition 2.10** For any graph $G = (V, E)$ with no isolated vertices and $|V| \geq 3$,

(i) $\gamma(G) \leq \gamma_s(G) \leq \gamma_{t,coi}(G)$.

(ii) $\gamma(G) \leq \gamma_t(G) \leq \gamma_{t,coi}(G)$.

**Proof.** Let $G = (V, E)$ be graph with no isolated vertices. Suppose that $S \subseteq V$ is any minimum total co-independent dominating set of $G$. Since for any graph $G$ any total co-independent dominating set $S$ is also split dominating set and every split dominating set is also dominating set. Hence $\gamma(G) \leq \gamma_s(G) \leq \gamma_{t,coi}(G)$. similarly we can proof (ii).

**Proposition 2.11** If $G = (V, E)$ is a graph with no isolated vertices and $|V| \geq 3$ and $H$ is spanning subgraph with no isolated vertices and has vertices greater than two of $G$ then, $\gamma_t(G) \leq \gamma_{t,coi}(H)$.
Proof. Let $S$ be any minimum total co-independent dominating set of $H$. Then obviously from the definition of the total co-independent domination, $S$ is also total dominating set of $H$. Therefore $S$ is total dominating set of $G$. Hence $\gamma_t(G) \leq \gamma_{t, coi}(H)$.

**Theorem 2.12** Let $G$ be a graph with $D$ as minimal total co-independent dominating set. then $V - D$ is independent dominating set of $G$.

**Proof.** Let $D$ be minimal total co-independent dominating set of $G$. Suppose that $V - D$ is not independent dominating set of $G$, since $D$ is total co-independent dominating set of $G$, then $V - D$ is independent set, that means if we suppose that $V - D$ is not independent dominating set of $G$, then there exists a vertex $u$ such that $u$ is not dominated by any vertex in $V - D$. Since $G$ has total co-independent dominating set, then $G$ has no isolated vertices, therefore $u$ is dominated by at least one vertex in $D - \{u\}$. Thus $D - \{u\}$ is total co-independent dominating set of $G$, which contradicts the minimality of $D$. Thus every vertex in $D$ is adjacent with at least one vertex in $V - D$ and $V - D$ is independent set. Hence $V - D$ is independent dominating set of $G$.

**Proposition 2.13** If $G = (V, E)$ is a graph with no isolated vertices and $|V| \geq 3$ and $H$ is spanning subgraph with no isolated vertices and has vertices greater than two of $G$ then, $\gamma_{t, coi}(H) \leq \gamma_{t, coi}(G)$.

**Proof.** As the number of independent vertices may increase in any connected spanning subgraph $H$ of $G$ we can still maximize the set $V - D$, which results in the decrease of the value $\gamma_{t, coi}$ of $G$. Hence, $\gamma_{t, coi}(H) \leq \gamma_{t, coi}(G)$.

**Observation 2.14** For any graph $G$ any total co-independent dominating set of $G$ contains all the support vertices.

**Proof.** Suppose the graph $G$ has a total co-independent dominating set $D$ and let $v$ be support vertex does not belongs to $D$ then clearly the pendent vertex which adjacent to $v$ can not belong to $D$ from the definition of total co-independent dominating set. Hence $D$ is not a dominating set, which is contradiction.

**Theorem 2.15** Let $G = (V, E)$ with total co-independent domination number $\gamma_{t, coi}(G)$, then

$$p - \beta \leq \gamma_{t, coi}(G) \leq q,$$

where $\beta$ is the independence number of $G$ and $q$ is the number of edges in $G$.

**Proof.** Since $V - D$ is independent in a total co-independent dominating set $D$ and since $\beta$ is the independence number, then obviously $p - \beta \leq \gamma_{t, coi}(G)$. 

For the upper bound we know that the total co-independent domination number exist if $G$ has no isolated vertices and has vertices more than two, therefore $q \geq p - 1$. Hence $\gamma_{t,coi}(G) \leq q$.

**Proposition 2.16** For any connected graph $G$, we have

$$\gamma_{t,coi}(G) \leq 2q - p + 1.$$ 

**Proof.** Since from Observation 2.4, $\gamma_{t,coi}(G) \leq p - 1$.

Hence $\gamma_{t,coi}(G) \leq p - 1 = 2(p - 1) - p + 1 \leq 2q - p + 1$.

**Theorem 2.17** For any graph $G$ with $p \geq 3$ with out isolated vertices and has complement graph $\overline{G}$ with out isolated vertices,

$$\gamma_{t,coi}(G) + \gamma_{t,coi}(\overline{G}) \leq \frac{p(p-1)}{2}.$$ 

**Proof.** From Theorem 2.15 we have $\gamma_{t,coi}(G) \leq q$ and similarly $\gamma_{t,coi}(\overline{G}) \leq \overline{q}$, where $\overline{q}$ is the number of edges in $\overline{G}$. Therefore $\gamma_{t,coi}(G) + \gamma_{t,coi}(\overline{G}) \leq q + \overline{q} = \frac{p(p-1)}{2} - p$. Hence $\gamma_{t,coi}(G) + \gamma_{t,coi}(\overline{G}) \leq \frac{p(p-1)}{2}$. The following result is obvious. Hence, we omit its proof.

**Theorem 2.18** For any graph $G$ with $p \geq 3$ with out isolated vertices and has complement graph $\overline{G}$ with out isolated vertices,

$$\gamma_{t,coi}(G)\gamma_{t,coi}(\overline{G}) \leq (p - 1)^2.$$ 

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**References**


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