Analysis of Unsteady State Heat Transfer in the Hollow Cylinder Using the Finite Volume Method with a Half Control Volume

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Abstract
Numerical methods are important tools for the heat transfer process simulation permitting the reproduction of experiments with cost and time reduction. The simple model simulation allows the validation of the computational scheme for future use in other projects with other geometries. This work presents the heat transfer problem in the cylinder heated from the central axis in an unsteady state, using the Finite Volume Method with a Half Control Volume. The radial temperature profile is shown for different Fourier numbers and different discretizations of the mesh, considering the Biot number and the inner radius pre-set.

Mathematics Subject Classification: 65N08

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1. Introduction

Heat transfer problems are prominent in engineering due to several applications in industry and environment performance of propulsion systems such as the design of conventional space and water heating systems, in the cooling of electronic equipment, in the design of refrigeration and air-conditioning systems, and in many manufacturing processes, as well plume and chemical nuclides dispersion, global warming, for example.

The main goal of the analysis of many transfer heat problems is to obtain the temperature profile within the system and transfer rate in certain conditions or, alternatively, the conditions needed, such as dimensions, shape, flow measurement, among others, to achieve a heat rate or temperature distribution, or both.

The simulation of heat transfer can be done computationally, the processes via numerical solution of the governing equations. This computer simulation is widely used in fundamental research and in industrial applications resulting in that, increasingly, new and better numerical methods are developed to improve their accuracy, efficiency, and range of applicability.

In [4] carried a comparative investigation of one-dimensional heat transfer problems in the heated hollow cylinder in the steady state using the Finite Difference Method and an alternative formulation called Finite Difference Method with Ghost Points, as well the Finite Volume Method and an alternative formulation known for Finite Volume Method with a Half Control Volume. This presents, on average, one to two orders of accuracy better when compared with other approaches, reason will be used in this work.

In order to find the numerical solution, the cylinder heated from the central axis, in an unsteady state problem, was treated using the Finite Volume Method with a Half Control Volume [3,9]. In the literature, this problem was investigated by [1,2,5,6,8].

2. Problem Formulation

2.1 Energy balance

The heat conduction in the hollow cylinder of internal radius \( r \) and height \( H \), can be evaluated by Fourier’s law (Figure 1):

\[
Q = -k(2\pi rH)\frac{dT}{dr} \tag{1}
\]

where \( k \) is the thermal conductivity (W/(m.K)), \( 2\pi rH \) is the cylinder surface area (m²) and \( dT/dr \) is the thermal gradient in the direction of the heat flow, which can expanded using Taylor series, like this
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Figure 1 – Geometry and control volume of the hollow cylinder heated from the central axis

\[ \dot{Q}_{r,\Delta r} = \left[ -k(2\pi rH) \frac{\partial T}{\partial r} \bigg|_{r=R_i} \right] + \Delta r \frac{\partial}{\partial r} \left[ -k(2\pi rH) \frac{\partial T}{\partial r} \bigg|_{r=R_i} \right] + O(\Delta r^2) . \]

From this,
\[ \dot{Q} = \dot{Q}_r - \dot{Q}_{r,\Delta r} = \left[ -k(2\pi rH) \frac{\partial T}{\partial r} \right] + \left[ k(2\pi rH) \frac{\partial T}{\partial r} \right] - \Delta r \frac{\partial}{\partial r} \left[ -k(2\pi rH) \frac{\partial T}{\partial r} \right] \Rightarrow \]
\[ \dot{Q} = \frac{\partial}{\partial r} \left( kr \frac{\partial T}{\partial r} \right) 2\pi H \Delta r + O(r^2) \quad (2) \]

Substituting the Eq. (2) in Eq. (1), one has,
\[ \frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \quad (3) \]

where \( \alpha = k/(\rho c) \) is the thermal diffusivity.

2.2 Boundary and initial conditions and nondimensionalization

If the internal and external radius of the cylinder are defined as \( R_i \) and \( R_e \), respectively, in the unsteady state, have the following boundary conditions:
\[ \left. \frac{\partial T}{\partial r} \right|_{r=R_i} = \frac{q_i}{2\pi k H R_i} \quad \text{for} \quad r = R_i \] and \[ \left. \frac{\partial T}{\partial r} \right|_{r=R_e} = -\frac{h}{k} \left[ T(R_e, t) - T_\infty \right] \quad \text{for} \quad r = R_e \] and, still, a

initial condition for \( t=0 \), being \( T(r,0) = T_0(r) = T_\infty \).

Defining the dimensionless variables with the asterisk as superscript, they can be written as \( r^* = r / R_e \), \( R_e^* = 1 \), \( R_i^* = R_i / R_e \), \( t^* = t \rho c R_e^2 / k \). Notice that \( t^* \) is the Fourier number, denoted by \( Fo \), which is a way of representing the dimensionless time. Thus, the Eq. (3) can be rewritten as
\[
\frac{1}{\alpha} \frac{\partial T^*}{\partial t} = \frac{1}{r^* R_c^* \partial r^*} \left( r^* R_c^* \frac{\partial T^*}{\partial r^*} \right) = \frac{\partial T^*}{\partial t} = \frac{1}{r^* \partial r^*} \left( r^* \frac{\partial T^*}{\partial r^*} \right)
\]

with the following boundary conditions for the \( r^* = R_i^* \) and \( r^* = R_e^* = 1 \), respectively,
\[
\frac{\partial T}{\partial r} \bigg|_{r^* = R_e^*} = R_i^* \frac{q'R_c}{2\pi R_i H} = -1 \quad \text{and} \quad \frac{\partial T}{\partial r} \bigg|_{r^* = 1} = -\frac{h R_e}{k} T^* = -Bi T^*, \quad \text{where} \quad Bi = \frac{h R_e}{k}
\]
is the Biot number. Rewritten the boundary conditions and omitting the asterisk in the dimensionless equations, we have
\[
\frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \quad (5a)
\]
\[
\frac{\partial T}{\partial r} \bigg|_{r = 0} = R_i = -1 \quad (5b)
\]
\[
\frac{\partial T}{\partial r} \bigg|_{r = 1} = 1 = -Bi T \quad (5c)
\]

3. Development of the algorithm

For the internal points, we discretize the Eq. (5-a) for the internal points, using the Finite Volume Method with a Half Control Volume. Considering the limits of integration in the control volume boundary being \( r = R_i + 1/2 \) and \( r = R_i - 1/2 \), we have:
\[
\int_{n}^{n+1} \int_{R_i}^{R_e} r \frac{\partial T}{\partial r} dr dt = \int_{n}^{n+1} r \left( \frac{\partial T}{\partial r} \right)_n - \left( \frac{\partial T}{\partial r} \right)_w \right] dt \quad (6)
\]

where \( e \) and \( w \) are the east and west faces, respectively, of control volumes in the numerical discretization. Applying the Crank-Nicolson method in the governing equation, we have:
\[
\left( -\frac{R_i \nu}{2} + \frac{\phi}{4} \right) T_{i-1}^{n+1} + (R_i + R_i \nu) T_i^{n+1} + \left( -\frac{R_i \nu}{2} - \frac{\phi}{4} \right) T_{i+1}^{n+1} = \left( \frac{R_i \nu}{2} - \frac{\phi}{4} \right) T_{i-1}^{n} + (R_i - R_i \nu) T_i^{n} + \left( \frac{R_i \nu}{2} + \frac{\phi}{4} \right) T_{i+1}^{n} \quad (7)
\]

Now, applying the Crank-Nicolson method in the boundary conditions for \( R_n \), we have:
\[
\int_{n}^{n+1} \int_{R_i}^{R_e} r \frac{\partial T}{\partial r} dr dt \Rightarrow \int_{n}^{n+1} \left[ \left( \frac{\partial T}{\partial r} \right)_e - \left( \frac{\partial T}{\partial r} \right)_w \right] \Delta t + \left[ \left( \frac{\partial T}{\partial r} \right)_n - \left( \frac{\partial T}{\partial r} \right)_w \right] \Rightarrow
\]
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\[ R_i(T_1^{n+1} - T_1^n) = \frac{1}{2} \left( \left( \frac{R_i \Delta t}{\Delta r^2} + \frac{\Delta t}{2 \Delta r} \right) (T_2^{n+1} - T_1^{n+1}) + \left( R_i \frac{\Delta t}{\Delta r} \right) \right) + \frac{1}{2} \left( \left( \frac{R_i \Delta t}{\Delta r^2} + \frac{\Delta t}{2 \Delta r} \right) (T_2^n - T_1^n) - \left( R_i \frac{\Delta t}{\Delta r} \right) \right) \]

(8)

and, taking \( \nu = \Delta t / \Delta r^2 \) and \( \phi = \Delta t / \Delta r \), we have

\[ \left( \frac{r_i + \frac{r_i \nu}{2} + \phi}{4} \right) T_1^{n+1} + \left( -\frac{r_i \nu}{2} - \frac{\phi}{4} \right) T_1^n = \left( \frac{r_i - \frac{r_i \nu}{2} - \frac{\phi}{4}}{4} \right) T_2^n + \left( \frac{r_i \nu}{2} + \phi \right) T_2^n + \phi R_i \]

(9)

Likewise, for \( R_e \):

\[ \int_0^r \int_0^l r \frac{\partial T}{\partial t} dr dt = \int_0^r \left[ \left( R_e \frac{\partial T}{\partial r} \right)^{n+1} - \left( R_w \frac{\partial T}{\partial r} \right)^n \right] dr \]

\[ \frac{R_e^2 - R_w^2}{2} \left( \frac{T_{m-1}^{n+1} - T_{m-1}^n}{\Delta t} \right) = \left( R_e \frac{\partial T}{\partial r} \right)_e^{n+1} - \left( R_w \frac{\partial T}{\partial r} \right)_w^n \]

(10)

and, being \( e = R_e \) and \( w = m \), we have

\[ \left( -\frac{r_m \nu}{2} + \phi \frac{T_m}{4} \right) T_{m-1}^{n+1} + \left( r_{m-1} + \frac{r_m \nu}{2} - \phi + \frac{Bi \phi}{2 + Bi \Delta r} \right) T_{m-1}^n = \]

\[ \left( -\frac{r_m \nu}{2} - \phi \frac{T_m}{4} \right) T_{m-2}^n + \left( r_{m-1} - \frac{r_m \nu}{2} + \phi - \frac{Bi \phi}{2 + Bi \Delta r} \right) T_{m-1}^n \]

(11)

4. Numerical Results

The results below show the temperature profile for different values for the Fourier number. In all cases, \( Bi=5 \), \( R_i=0.15 \), \( R_e=1 \), \( \Delta t = 0.001 \) and grid discretizations with 10, 20, 50 and 100 points were adopted.

For the methodology proposed, was calculated the mean square error \([7]\),

\[ e = \sqrt{\frac{1}{n+1} \sum_{i=1}^{n+1} \varepsilon_i^2} \]

where \( \varepsilon_i = T_i - T_i^* \), with \( T_i \) the numerical solution and \( T_i^* \) the steady state solution, both in the point \( i \), with \( Bi = 5 \), \( R_i = 0.15 \), \( R_e = 1 \) and \( Fo=5 \).

The Tab. 1 shows the error associated for the numerical approximation considering different grid discretizations.

5. Conclusions

The temperature profile of the cylinder heated from the central axis in an unsteady state, using the Finite Volume Method with a Half Control Volume was
calculated for different Fourier numbers and different discretizations of the mesh, considering the Biot number and the inner radius pre-set.

![Figure 1- Temperature profile for grid with (a) 10 and (b) 20 points and Δt=0.001.](image1)

![Figure 2- Temperature profile for grid with (a) 50 and (b) 100 points and Δt=0.001.](image2)

The Finite Volume Method with *Half* Volume control showed good accuracy. Besides, its formulation is easily implemented and presented, as a final result, a matrix with only three nonzero diagonals. This advantage can be exploited to store only the coefficients of the three diagonals, which carries a low computational cost and reduced computational time. Besides, the simple model simulation allows the validation of the computational scheme for future use in other projects with other geometries.
Table 1: Mean square error associated for the numerical approximation.

<table>
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<tr>
<th>number of mesh points</th>
<th>$\Delta r$</th>
<th>Mean square error</th>
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<tr>
<td>10</td>
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<td>0.32E-02</td>
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<tr>
<td>20</td>
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<td>0.8E-02</td>
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<td>400</td>
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<td>9.52E-05</td>
</tr>
<tr>
<td>500</td>
<td>0.0017</td>
<td>7.62E-05</td>
</tr>
</tbody>
</table>

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References


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