A Summation Formula for the Multivariable A-Function

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Abstract

In this paper we establish a summation formula for the A-function of ‘r’ variables defined by Gautam G.P. and Goyal A.N. Three particular cases of this formula have also been discussed. In one particular case our formula reduces to a result involving multivariable A-function and Jacobi polynomial. Another particular case gives rise to a new formula involving multivariable A-function and Chebyshev polynomial. On account of the most general nature of the function involved herein, the summation formula which we presented here can indeed be applied in order to obtain a large number of new interesting and useful results.

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1 Introduction

The A-function of ‘r’ variables defined by Gautam G.P. and Goyal A.N.[2] by means of multiple Mellin-Barnes type contour integral is represented here in the following form [9,p.197]:
$$A[z_1, ..., z_r] = A_{P,Q}(m_r, n_r) \left( \begin{array}{c}
\sum_{i=1}^{r} (a_j; \alpha_j^{(1)}, ..., \alpha_j^{(r)})_{1,P} : ((c_j^{(i)}, \gamma_j^{(i)})_{1,p_i}) \\
\sum_{i=1}^{r} (b_j; \beta_j^{(1)}, ..., \beta_j^{(r)})_{1,Q} : ((d_j^{(r)}, \delta_j^{(r)})_{1,q_r}) 
\end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} ... \int_{L_r} \phi(s_1, ..., s_r) \left[ \prod_{i=1}^{r} \theta_i(s_i)^{s_i} \right] ds_1 ... ds_r \quad (1.1)$$

where \(\omega = \sqrt{-1}\) and

$$\phi(s_1, ..., s_r) = \prod_{j=1}^{M} \Gamma \left( b_j - \sum_{i=1}^{r} \beta_j^{(i)} s_i \right) \prod_{j=1}^{N} \Gamma \left( 1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i \right)$$

$$\prod_{j=M+1}^{Q} \Gamma \left( 1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} s_i \right) \prod_{j=N+1}^{P} \Gamma \left( a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i \right) \quad (1.2)$$

$$\theta_i(s_i) = \prod_{j=1}^{m_i} \Gamma \left( d_j^{(i)} - \delta_j^{(i)} s_i \right) \prod_{j=1}^{n_i} \Gamma \left( 1 - c_j^{(i)} + \gamma_j^{(i)} s_i \right)$$

$$\prod_{j=m_i+1}^{q_i} \Gamma \left( 1 - d_j^{(i)} + \delta_j^{(i)} s_i \right) \prod_{j=n_i+1}^{p_i} \Gamma \left( c_j^{(i)} - \gamma_j^{(i)} s_i \right) \quad (1.3)$$

Here \(M, N, P, Q, m_i, n_i, p_i, q_i\) are non-negative integers with \(M \leq Q, N \leq P,\)

\(m_i \leq q_i\) and \(n_i \leq p_i\

and the parameters \(z_i(\neq 0), a_j, b_j, \alpha_j^{(i)}, \beta_j^{(i)}, c_j^{(i)}, d_j^{(i)}, \gamma_j^{(i)}, \delta_j^{(i)}\)

are all complex numbers for all \(i = 1, 2, ..., r\). Further \((m_r, n_r)\) stands for

\(m_1, m_2, ..., m_r, n_r\) and \((p_r, q_r)\) stands for \(p_1, q_1, ..., p_r, q_r\). Also \((a_j; \alpha_j^{(1)}, ..., \alpha_j^{(r)})_{1,P}\)

and \(((c_j^{(i)}, \gamma_j^{(i)})_{1,p_i})\) abbreviate the parameter sequences \((a_i; \alpha_i^{(1)}, ..., \alpha_i^{(r)})\),

\((a_p; \alpha_p^{(1)}, ..., \alpha_p^{(r)})\) and \(((c_j^{(i)}, \gamma_j^{(i)})_{1,p_i})\) respectively, and similar

representation hold for \((b_j; \beta_j^{(1)}, ..., \beta_j^{(r)})_{1,Q},\) \(((d_j^{(i)}, \delta_j^{(i)})_{1,q_i})\) and so on.

Integral (1.1) is absolutely convergent if \(\xi_i^* = 0, \eta_i > 0\) and \(|arg(\xi_i)z_i| < \frac{\pi}{2}\eta_i\)

where

$$\xi_i = \left( \prod_{j=1}^{P} \left\{ \alpha_j^{(i)} \right\}^{a_j^{(i)}} \prod_{j=1}^{Q} \left\{ \beta_j^{(i)} \right\}^{-b_j^{(i)}} \prod_{j=1}^{q_i} \left\{ \delta_j^{(i)} \right\}^{-\delta_j^{(i)}} \prod_{j=1}^{n_i} \left\{ \gamma_j^{(i)} \right\}^{\gamma_j^{(i)}} \right)$$

$$\xi_i^* = Im \left( \sum_{j=1}^{P} \alpha_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{q_i} \delta_j^{(i)} - \sum_{j=1}^{n_i} \gamma_j^{(i)} \right) \quad (1.5)$$

$$\eta_i = Re \left( \sum_{j=1}^{N} \alpha_j^{(i)} - \sum_{j=N+1}^{P} \alpha_j^{(i)} + \sum_{j=1}^{M} \beta_j^{(i)} - \sum_{j=M+1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} \right) \quad (1.6)$$
When $M = 0$ and $\alpha_j^{(i)}, \beta_j^{(i)}, \gamma_j^{(i)}, \delta_j^{(i)}$ are all positive real numbers, (1.1) reduces to the $H$-function of $'r'$ variables given by Srivastava and Panda [11,12]. For details about the nature of contours $L_1, ..., L_r$ and other special cases of the multivariable $A$-function, we refer to the papers [2,8,9].

2 Results used

To facilitate the derivation of our main result, we shall require the following results.

(a) Richard Askey[7]:

$$(\sin \theta)^{1-2u} P_n^{1-u}(\cos \theta) = \sum_{k=0}^{\infty} \frac{2^{2u} (n+k)! \Gamma(n+2-2u) \Gamma(k+u) \sin(n+2k+1)\theta}{\Gamma(1-u) \Gamma(u) k! n! \Gamma(n+k+2-u)}$$

provided $u < 1$, $0 \leq \theta \leq \pi$ and 'n' is a non negative integer.

(b) Rainville [6]:

$$\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma \left(z + \frac{1}{2}\right)$$

and

$$(i) \quad \alpha_n^\mu = \sum_{m=0}^{n} \frac{(2\mu)^{n+m} (z-1)^m}{2^m m! (n-m)! (\mu + \frac{1}{2})^m}$$

where $(a)_n$ is the pochhammer symbol defined by,

$(a)_n = a(a+1)(a+2)....(a+n-1)$ and $(a)_0 = 1$

(c) Srivastava , Gupta and Goyal [10]:

$$(i) \quad P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[-n, 1+\alpha + \beta + n ; 1+\alpha ; \frac{1-x}{2}\right]$$

provided $\text{Re}(\alpha + 1) > 0$, $\text{Re} (\beta + 1) > 0$; where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial.

$$(ii) \quad U_n(x) = \frac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)} ; \ n \geq 0$$

where $U_n(x)$ represents the Chebyshev polynomial of second kind.

and

$$(iii) \quad U_n(x) = \frac{(n+1)!}{\left(\frac{3}{2}\right)_n} P_n^{\left(\frac{3}{2},\frac{1}{2}\right)}(x)$$

(d) Erdelyi [1]:

$$\sin n z = n \sin z {}_2F_1 \left[\frac{1-n}{2}, \frac{1+n}{2} ; \frac{3}{2} ; \sin^2 z\right] ; |\sin z| < 1$$
3 Main Result

\[ \frac{\sqrt{\pi}}{2} (\sin \theta)^{2\nu - 1} \sum_{m=0}^{n} \left[ \frac{(\cos \theta - 1)^m}{2^m m! (n-m)!} \right] \times \]

\[ A_{M+1,N: (m_r, n_r)}^{P+1, Q+1: (p_r, q_r)} \left[ \begin{array}{c}
\frac{z_1}{\sin^{\nu + \theta}} \\
\vdots \\
\frac{z_r}{\sin^{\nu + \theta}} \\
\end{array} \right] \]

\[ = \sum_{k=0}^{\infty} \left\{ \frac{(n+k)!}{n! k!} \sin(n+2k+1) \theta \right\} \times \]

\[ \left[ \begin{array}{c}
\nu - k; \lambda_1, ..., \lambda_r, (a_j; \alpha_j^{(1)}), ..., (a_j^{(r)}); \alpha_j^{(1)}; 1, P, \lambda_1, ..., \lambda_r, (c_j; \gamma_j^{(1)}); \lambda_1, ..., \lambda_r, (d_j^{(i)}); \delta_j^{(i)}; 1, q_i \\
\end{array} \right] \]

provided 'n' is a non-negative integer; \( \lambda_i \geq 0, i = 1, 2, ..., r \) (all \( \lambda_i \) are not simultaneously zero); \( 0 \leq \theta \leq \pi; |\arg z_i| < \frac{1}{2} \pi \eta_i, \eta_i > 0; \) where \( \eta_i \) are given by (1.6), \( i = 1, 2, ..., r \) and \( \text{Re} (2\nu - 1) \geq 0 \).

4 Proof

Replacing ‘u’ by \( (1 - \nu + \sum_{i=1}^{r} \lambda_i s_i) \), (2.1) takes the form:

\[ (\sin \theta)^{2\nu - 1 - 2(\sum_{i=1}^{r} \lambda_i s_i)} \times P_{n}^{(\nu - (\sum_{i=1}^{r} \lambda_i s_i))} (\cos \theta) \]

\[ = 2^{2(1-\nu+\sum_{i=1}^{r} \lambda_i s_i)} \frac{\Gamma \left( n + 2\nu - 2 \left( \sum_{i=1}^{r} \lambda_i s_i \right) \right)}{\Gamma \left( \nu - \sum_{i=1}^{r} \lambda_i s_i \right)} \times \]

\[ \sum_{k=0}^{\infty} \left\{ \frac{(n+k)!}{n! k!} \Gamma (k + 1 - \nu + \sum_{i=1}^{r} \lambda_i s_i) \sin(n+2k+1) \theta \right\} \]

provided \( \left( \nu - \sum_{i=1}^{r} \lambda_i s_i \right) > 0, 0 \leq \theta \leq \pi \) and ‘n’ is a non-negative integer.

To prove (3.1), multiply (4.1) by the expression,
the integral is absolutely convergent when the given conditions are satisfied. Replace $P^r_0(\nu - \sum_{i=1}^r \lambda_i s_i)(\cos \theta)$ by the series given by (2.3) and integrate ‘r’ times along the contours $L_1, L_2, ..., L_r$ defined in (1.1) to get

\[
\frac{1}{(2\pi \omega)^r} \left( \frac{\sin \theta}{2} \right)^{2\nu - 1} \int_{L_1} \ldots \int_{L_r} \phi(s_1, ..., s_r) \prod_{i=1}^r \theta_i(s_i) \left( \frac{z_i}{\sin 2\lambda_i \theta} \right)^{s_i} \times 
\sum_{m=0}^{n} \frac{(\cos \theta - 1)^m}{2^m m! (n-m)!} \Gamma \left( \frac{1}{2} + m + \nu - \sum_{i=1}^r \lambda_i s_i \right) ds_1 \ldots ds_r
\]

\[
= \frac{2^{(1-\nu)}}{(2\pi \omega)^r} \int_{L_1} \ldots \int_{L_r} \phi(s_1, ..., s_r) \prod_{i=1}^r \theta_i(s_i) z_i^{s_i} \Gamma \left( \frac{n + 2\nu - 2 \sum_{i=1}^r \lambda_i s_i}{1 - \nu + \sum_{i=1}^r \lambda_i s_i} \right) \Gamma \left( 1 - \nu + \sum_{i=1}^r \lambda_i s_i \right) \frac{1}{n!} \times 
\sum_{k=0}^\infty \frac{(n+k)!}{k!} \frac{(1 - \nu + k + \sum_{i=1}^r \lambda_i s_i)}{\sin(n+2k+1) \theta} \sin(n+2k+1) \theta ds_1 \ldots ds_r
\]

Changing the order of integration and summation and using (1.1) the result follows. On the left hand side the change in the order of integration and summation is justified because the series is finite and integrals exist. On the right hand side it is justified because the series converges uniformly with respect to $s_1, s_2, ..., s_r$ and $\phi(s_1, ..., s_r) \prod_{i=1}^r (\theta_i(s_i) z_i^{s_i}) \frac{1}{\Gamma \left( 1 - \nu + \sum_{i=1}^r \lambda_i s_i \right) n!}$ is continuous and the integral is absolutely convergent when the given conditions are satisfied.

5 Special Cases

Case 1.: When $n$ is even (say $n = 2\ell$) the summation formula (3.1) takes the form:

\[
\sum_{m=0}^{2\ell} \frac{(\cos \theta - 1)^m}{2^m m! (2\ell - m)!} \times 
A_{P+1,Q+1}^{M+1,N;\{m_r,s_r\}} \frac{z_1 \sin^{s_1+1} \theta}{\sin^{s_1+1} \theta} : \ldots : 
\frac{z_r \sin^{s_r+1} \theta}{\sin^{s_r+1} \theta}
\]

\[
= \Omega \times A_{P+2,Q+2}^{M+1,N+1;\{m_r,s_r\}} \frac{z_1}{\sin \theta} : \ldots : 
\frac{z_r}{\sin \theta}
\]

\[
\begin{align*}
&\frac{\Gamma(n+2\nu-2\sum_{i=1}^r \lambda_i s_i)}{\Gamma(1-\nu+\sum_{i=1}^r \lambda_i s_i) n!} \times \\
&\sum_{k=0}^\infty \frac{(n+k)!}{k!} \frac{(1-\nu+k+\sum_{i=1}^r \lambda_i s_i)}{\sin(n+2k+1) \theta} \sin(n+2k+1) \theta ds_1 \ldots ds_r
\end{align*}
\]
where either
\[
\Omega = 2(sinθ)^2 - 2ν \sum_{k=0}^{∞} \frac{(2k + 1)!}{(2k)! k! (2ν + k + 1)} P_{ν+k}^{(ν+k+1)}(cosθ)
\]
or
\[
\Omega = \frac{2}{√π} (sinθ)^2 - 2ν \sum_{k=0}^{∞} \frac{(2k + 1)!}{k! (2ν + k + 1)} U_{2ν+2k}(cosθ)
\]
provided the validity conditions of (3.1) with \(n = 2ν\) holds.

**Case 2:** When \(n\) is odd (say \(n = 2ν + 1\)) the summation formula (3.1) takes the form:
\[
\sum_{m=0}^{2ν+1} \left[ \frac{(cosθ - 1)^m}{2^m m!(2ν + 1 - m)!} \right] \times
\]
\[
\begin{align*}
A_{P+1,Q+1:(p,q)}^{M+1,N+1:(r,s)} & \left[ \begin{array}{c}
\frac{z_1}{sin^2θ_1} \\
\vdots \\
\frac{z_r}{sin^2θ_r}
\end{array} \right] \\
& \begin{array}{c}
(a_j; α_j^{(1)}, ..., α_j^{(r)})_{1,p}, (ν + m; λ_1, ..., λ_r) : ((c_j^{(i)}, γ_j^{(i)})_{1,q}) \\
(2ν + m + 2k + 2; λ_1, ..., 2ν + 2k + 2); (b_j; β_j^{(1)}, ..., β_j^{(r)})_{1,q} : ((d_j^{(i)}, δ_j^{(i)})_{1,q})
\end{array}
\end{align*}
\]
for any
\[
\Omega = \frac{2}{√π} (sinθ)^2 - 2ν \sum_{k=0}^{∞} \frac{(2ν + k + 1)!}{k!(2ν + k + 1)!} U_{2ν+2k+1}(cosθ)
\]
or
\[
\Omega = 2(sinθ)^2 (sinθ)^{1 - 2ν} \sum_{k=0}^{∞} \frac{(2ν + k + 1)!}{k!(2ν + k + 1)!} U_{ν+k+1}(cosθ)
\]
provided the validity conditions of (3.1) with \(n = 2ν + 1\) holds.

**Case 3:** When \(M = 0\) and \(α_j^{(i)}, β_j^{(i)}, γ_j^{(i)}, δ_j^{(i)}\) are all positive real numbers in (3.1), (5.1) and (5.2) then these formulae reduce to the corresponding formulae involving H-function of \('r'\) variables defined by Srivastava[11,12]. Further when \(r=2\), these formulae reduce to the result obtained by Vasudevan Nambisan [13].
References


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