Exact Solutions for a Generalized KdV Equation with Time-Dependent Coefficients and $K(m, n)$ Equation

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Abstract

In this paper, a generalized KdV equation with time-dependent coefficients will be studied. The $K(m, n)$ equation with generalized evolution will also be examined. The Riccati equation mapping method will be used to obtain some new exact solutions for both equations. These solutions include solitary wave solutions and periodic wave solutions. The results presented in this paper improve the previous results.

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1 Introduction

Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of science, especially in physics. Searching for exact soliton solutions of NPDEs plays an important and significant role in the study on the dynamics of those phenomena. Up to now, many effective ansatz methods have been presented, such as the tanh method [1], Jacobi elliptic function method [2], F-expansion method [3], the Exp-function method [4-7], auxiliary equation method [8,9], and so on. Here, it is worth to mention that the Riccati equation mapping method [10,11]. The solution procedure of this method, with the aid of Maple, is of utter simplicity and this method can easily extented to other kinds of nonlinear evolution equations.

In this work, by using the Riccati equation mapping method, we aim to investigate two equations. First we will examine the generalized KdV equation
which was given by Wazwaz [12]

\[ w_t + f(t)w^n w_x + g(t)w_{xxx} = 0, \quad n \geq 3, \]  

(1)

where \( f(t) \) and \( g(t) \) are time-dependent coefficients. If we set \( f(t) = 6, \ g(t) = 1, \) and \( n = 1, 2, \) Eq. (1) will give the standard KdV equation and mKdV equation, respectively. It is well known that the KdV equation and the mKdV equation are integrable, whereas the generalized KdV equation (1) is non integrable for \( n \geq 3. \)

The second model that will be examined is the \( K(m, n) \) equation with generalized evolution which was given by Biswas [13]

\[ (u')_t + au^m u_x + b(u^n)_{xxx} = 0, \]  

(2)

where \( a, b \in \mathbb{R} \) are constants, while \( l, m, n \in \mathbb{Z}^+ \). Generally, the \( k(m, n) \) equation (2) is non-integrable.

For Eqs. (1) and (2), using a solitary wave ansatz in the form of \( \text{sech}^p \) function, Wazwaz[12] and Biswas[13] obtained the 1-soliton solution for the equations (1) and (2), respectively. In this paper, we will explore more new exact solutions for Eqs. (1) and (2).

2 Description of the method

In this section, we review the Riccati equation mapping method [10,11] at first.

Given a nonlinear partial differential equation, for instance, in two variables, as follows:

\[ P(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0, \]  

(3)

where \( P \) is in general a nonlinear function of its variables.

We firstly use the Exp-function method to obtain new exact solutions of the following Riccati equation

\[ \phi' = \frac{d}{d\xi} \phi = A + \gamma \phi^2, \]  

(4)

where \( A \) and \( \gamma \) are arbitrary constants, then using the Riccati equation (4) as auxiliary equation and its exact solutions, we obtain exact solutions of the nonlinear partial differential equation(3).

Seeking for the exact solutions of Eq. (4), we introducing a complex variable \( \eta \), defined by

\[ \eta = \rho \xi + \xi_0, \]  

(5)

where \( \rho \) is a constant to be determined later, \( \xi_0 \) is an arbitrary constant, Riccati equation (4) converts to

\[ \rho \phi' - A - \gamma \phi^2 = 0, \]  

(6)
where prime denotes the derivative with respect to $\eta$.

According to the Exp-function method, we assume that the solution of Eq. (6) can be expressed in the following form

$$
\phi(\eta) = \frac{a_e \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_g \exp(g\eta) + \cdots + b_{-f} \exp(-f\eta)},
$$

(7)

where $e$, $d$, $g$ and $f$ are positive integers which are given by the homogeneous balance principle, $a_e, \ldots, a_{-d}$, $b_g, \ldots, b_{-f}$ are unknown constants to be determined. To determine the values of $e$ and $g$, we usually balance the linear term of the highest order in Eq. (6) with the highest order nonlinear term. Similarly, we can determine $d$ and $f$ by balancing the linear term of the lowest order in Eq. (6) with the lowest order nonlinear term, we obtain $e = g$, $d = f$.

For simplicity, we set $e = g = 1$ and $d = f = 1$, then Eq. (7) becomes

$$
\phi(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)},
$$

(8)

Substituting Eq. (8) into Eq. (6), equating to zero the coefficients of all powers of $\exp(n\eta)$ ($n = -2, -1, 0, 1, 2$) yields a set of algebraic equations for $a_1$, $a_0$, $a_{-1}$, $b_1$, $b_0$, $b_{-1}$ and $\mu$. Solving the system of algebraic equations by using Maple, we obtain the new exact solution of Eq. (4), which read

$$
\phi_1 = -\sqrt{-\frac{4}{\gamma}}b_1 \exp(\gamma \sqrt{-\frac{4}{\gamma}}\xi + \xi_0) + a_{-1} \exp(-\gamma \sqrt{-\frac{4}{\gamma}}\xi - \xi_0) + a_0 \exp(\gamma \sqrt{-\frac{4}{\gamma}}\xi + \xi_0) + a_{-1} \exp(-\gamma \sqrt{-\frac{4}{\gamma}}\xi - \xi_0),
$$

(9)

where $a_{-1}$ and $b_1$ are free parameters;

$$
\phi_2 = \frac{(\gamma a_0^2 + A b_0^2)}{4\gamma \sqrt{-\frac{4}{\gamma}}b_{-1}} \exp(2\gamma \sqrt{-\frac{4}{\gamma}}\xi + \xi_0) + a_0 + \sqrt{-\frac{4}{\gamma}}b_{-1} \exp(-2\gamma \sqrt{-\frac{4}{\gamma}}\xi - \xi_0)
$$

$$
\frac{(\gamma a_0^2 + A b_0^2)}{4\gamma \sqrt{-\frac{4}{\gamma}}b_{-1}} \exp(2\gamma \sqrt{-\frac{4}{\gamma}}\xi + \xi_0) + b_0 + b_{-1} \exp(-2\gamma \sqrt{-\frac{4}{\gamma}}\xi - \xi_0),
$$

(10)

where $a_0$, $b_0$ and $b_{-1}$ are free parameters.

By choosing properly values of $a_0$, $a_{-1}$, $b_0$, $b_{-1}$, we find many kinds of hyperbolic function solutions and triangular periodic solutions of Eq. (4), which are listed as follows:

(i) When $\xi_0 = 0$, $b_1 = 1$, $a_{-1} = \pm \sqrt{-\frac{4}{\gamma}}\frac{A}{\gamma} < 0$, the solution (9) becomes

$$
\phi = -\sqrt{-\frac{A}{\gamma}} \tanh(\gamma \sqrt{-\frac{A}{\gamma}}\xi),
$$

(11)
and

\[ \phi = -\sqrt{\frac{A}{\gamma}} \coth(\gamma \sqrt{\frac{A}{\gamma}}\xi). \] (12)

(ii) When \( \xi_0 = 0, b_1 = i, a_{-1} = \pm \sqrt{\frac{A}{\gamma}}, \frac{A}{\gamma} > 0, \) the solution (9) becomes

\[ \phi = \sqrt{\frac{A}{\gamma}} \tan(\gamma \sqrt{\frac{A}{\gamma}}\xi), \] (13)

and

\[ \phi = -\sqrt{\frac{A}{\gamma}} \cot(\gamma \sqrt{\frac{A}{\gamma}}\xi). \] (14)

(iii) When \( \xi_0 = 0, b_0 = 0, b_{-1} = 1, a_0 = \pm 2\sqrt{\frac{A}{\gamma}}, \frac{A}{\gamma} < 0, \) the solution (10) becomes

\[ \phi = -\sqrt{\frac{A}{\gamma}} \left[ \coth(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \pm \text{csch}(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \right]. \] (15)

(iv) When \( \xi_0 = 0, b_0 = 0, b_{-1} = i, a_0 = \pm 2\sqrt{\frac{A}{\gamma}}, \frac{A}{\gamma} < 0, \) the solution (10) becomes

\[ \phi = -\sqrt{\frac{A}{\gamma}} \left[ \tanh(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \pm i \text{sech}(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \right]. \] (16)

(v) When \( \xi_0 = 0, b_0 = 0, b_{-1} = 1, a_0 = \pm 2\sqrt{\frac{A}{\gamma}}, \frac{A}{\gamma} > 0, \) the solution (10) becomes

\[ \phi = \sqrt{\frac{A}{\gamma}} \left[ \tan(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \right]. \] (17)

(vi) When \( \xi_0 = 0, b_0 = 0, b_{-1} = i, a_0 = \pm 2\sqrt{\frac{A}{\gamma}}, \frac{A}{\gamma} > 0, \) the solution (10) becomes

\[ \phi = -\sqrt{\frac{A}{\gamma}} \left[ \cot(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \mp \csc(2\gamma \sqrt{\frac{A}{\gamma}}\xi) \right]. \] (18)

For simplicity, in the rest of the paper, we consider \( \xi_0 = 0. \)

3 The exact solutions of Eq. (1)

In this section, using the Riccati equation (4) as auxiliary equation and its the exact traveling wave solutions, we investigate Eq. (1) and derive the various exact solutions of Eq. (1). Balancing the order of the nonlinear term \( w^n w_x \) with the term \( w_{xxx} \) in (1), we obtain

\[ nP + P + 1 = P + 3, \]
so that
\[ P = \frac{2}{n}. \]

To get a closed form solution, it is natural to use the transformation
\[ w = v^{\frac{1}{n}}, \]
we have
\[ n^2 v^2 v_t + f(t)n^2 v^3 v_x + g(t)[(1-n)(1-2n)(v_x)^3 + 3n(1-n)vv_x v_{xx} + n^2 v^2 v_{xxx}] = 0. \]

In order to obtain new exact travelling wave solutions for Eq. (20), we use
\[ v(x, t) = v(\xi), \quad \xi = k(t)(x - c(t)t), \]
where \( k(t) \) and \( c(t) \) are functions in \( t \) to be determined later, and substituting (21) into Eq. (20), we obtain
\[ n^2 v^2 v_t + k(t)f(t)n^2 v^3 v' + g(t)[(1-n)(1-2n)k^3(t)(v')^3 + 3n(1-n)k^3(t)vv'v'' + n^2 k^3(t)v^2 v'''] = 0. \]

Now, we assume that the solution of Eq. (22) can be expressed in the following form
\[ v = v(\xi) = \sum_{j=0}^{N} \alpha_j(t)\phi^j(\xi) + \sum_{j=1}^{N} \beta_j(t)\phi^{-j}(\xi), \]
where \( N \) is positive integers which are given by the homogeneous balance principle, \( \alpha_j(t)(j = 0, 1, \cdots, N) \) and \( \beta_j(t)(j = 1, \cdots, N) \) are functions in \( t \) to be determined later, \( \phi(\xi) \) is a solution of the Riccati Eq. (4). Balancing \( v^2 v''' \) term with \( v^3 v' \) term in (22) gives \( N = 2 \). Therefore, we obtain
\[ v = \alpha_0(t) + \alpha_1(t)\phi(\xi) + \alpha_2(t)\phi^2(\xi) + \frac{\beta_1(t)}{\phi(\xi)} + \frac{\beta_2(t)}{\phi^2(\xi)}. \]

Substituting Eq. (24) into (22) and using the Riccati equation (4), equating to zero the coefficients of all powers of \( \phi(\xi) \) yields a set of first-order ODEs and algebraic equations for \( \alpha_0(t) \), \( \alpha_1(t) \), \( \alpha_2(t) \), \( \beta_1(t) \), \( \beta_2(t) \), \( k(t) \) and \( c(t) \), solving the resulting system we find the following sets of solutions.

Case 1
\[ k(t) = k, \quad \alpha_1(t) = 0, \quad \beta_1(t) = 0, \]
\[ \alpha_0(t) = \lambda_0, \quad \alpha_2(t) = \lambda_2, \quad \beta_2(t) = 0, \quad A = \frac{\lambda_0}{\lambda_2} \gamma, \]
we obtain
\[ c(t) = -\frac{4k^2\gamma^2\lambda_0}{tn^2\lambda_2} \int g(t)dt + \frac{C_1}{t}, \quad f(t) = -\frac{2(n+1)(n+2)g(t)k^2\gamma^2}{n^2\lambda_2}, \tag{25} \]

where \( k, \lambda_0, \lambda_2 \) and \( C_1 \) are arbitrary constants.

Case 2

\[ k(t) = k, \quad \alpha_1(t) = 0, \quad \beta_1(t) = 0, \]
\[ \alpha_0(t) = \mu_0, \quad \alpha_2(t) = 0, \quad \beta_2(t) = \mu_2, \quad A = \frac{\mu_2}{\mu_0}\gamma, \]

\[ c(t) = -\frac{4k^2\gamma^2\mu_2}{tn^2\mu_0} \int g(t)dt + \frac{C_2}{t}, \quad f(t) = -\frac{2(n+1)(n+2)g(t)k^2\gamma^2\mu_2}{n^2\mu_0^2}, \tag{26} \]

where \( k, \mu_0, \mu_2 \) and \( C_2 \) are arbitrary constants.

Thus from Eqs. (24), (25) and (26), we obtain families of exact solutions to Eq. (22) as follows,
\[ v = \lambda_0 + \lambda_2 \phi^2(\xi), \tag{27} \]
\[ v = \mu_0 + \mu_2 \frac{1}{\phi^2(\xi)}, \tag{28} \]

where \( \phi(\xi) \) is a solution of the Riccati Eq. (4).

Substituting solutions (11) - (18) of the Riccati Eq. (4) into (27) and (28), using the transformation (19), we have the following several families of solutions to Eq. (1).

**Family 1**

\[ w_1(x, t) = \left\{ \lambda_0 + \lambda_2 \left[ -\frac{\lambda_0}{\lambda_2} b_1 \exp(\gamma \sqrt{-\frac{\lambda_0}{\lambda_2} \xi}) + a_{-1} \exp(-\gamma \sqrt{-\frac{\lambda_0}{\lambda_2} \xi}) \right] \right\}^\frac{1}{n}, \tag{29} \]

where \( \xi = k \left[ x + \frac{4k^2\gamma^2\lambda_0}{n^2\lambda_2} \int g(t)dt - C_1 \right], \quad f(t) = -\frac{2(n+1)(n+2)g(t)k^2\gamma^2}{n^2\lambda_2}. \]

If we set \( b_1 = 1, \ a_{-1} = \pm \sqrt{-\frac{A}{\gamma}} = \pm \sqrt{-\frac{\lambda_0}{\lambda_2}}, \) and \( A = \frac{\lambda_0}{\gamma} < 0 \) in Eq. (29), we obtain
\[ w_{1(1)}(x, t) = \left[ \lambda_0 \sech^2(\gamma \sqrt{-\frac{\lambda_0}{\lambda_2} \xi}) \right]^\frac{1}{n}, \tag{30} \]

and
\[ w_{1(2)}(x, t) = \left[ -\lambda_0 \csch^2(\gamma \sqrt{-\frac{\lambda_0}{\lambda_2} \xi}) \right]^\frac{1}{n}. \tag{31} \]
Setting $b_1 = i$, $a_{-1} = \mp \sqrt{\frac{4}{\gamma}} = \mp \sqrt{\frac{2m}{\lambda_2}}$, and $\frac{4}{\gamma} = \frac{2m}{\lambda_2} > 0$ in Eq. (29), we get
\[ w_{1(3)}(x, t) = \left[ \lambda_0 \sec^2(\gamma \sqrt{\frac{\lambda_0}{\lambda_2}}) \right]^\frac{1}{n}, \tag{32} \]
and
\[ w_{1(4)}(x, t) = \left[ \lambda_0 \csc^2(\gamma \sqrt{\frac{\lambda_0}{\lambda_2}}) \right]^\frac{1}{n}. \tag{33} \]

**Family 2**

\[ w_2(x, t) = \left\{ \begin{array}{c} \lambda_0 + \lambda_2 \left[ \frac{(\gamma n^2 + \lambda_0^2)}{4 \gamma n^2 - \lambda_0^2} \exp(2\eta) + b_0 + b_{-1} \exp(-2\eta) \right]^{\frac{n}{2}} \\
\frac{(\gamma n^2 + \lambda_0^2)}{4 Ab_{-1}} \exp(2\eta) + b_0 - b_{-1} \exp(-2\eta) \end{array} \right\}, \tag{34} \]

where $\xi = k \left[ x + \frac{4k^2 + 2\lambda_n}{\gamma n^2} \int g(t) dt - C_1 \right]$, $f(t) = -\frac{2(n+1)(n+2)\gamma k^2 \gamma^2}{n^2 \lambda_2}$, $A = \frac{\lambda m \gamma}{\lambda_2}$, and $\eta = \gamma \sqrt{-\frac{\lambda}{\lambda_2}}$

If we set $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{-\frac{k}{\gamma}} = \pm 2\sqrt{-\frac{2m}{\lambda_2}}$, and $\frac{4}{\gamma} = \frac{2m}{\lambda_2} < 0$ in Eq. (34), we obtain
\[ w_{2(1)}(x, t) = \left[ \frac{2\lambda_0}{1 \mp \cosh(2\gamma \sqrt{-\frac{2m}{\lambda_2}})} \right]^\frac{1}{n}. \tag{35} \]

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{k}{\gamma}} = \pm 2\sqrt{-\frac{2m}{\lambda_2}}$, and $\frac{4}{\gamma} = \frac{2m}{\lambda_2} < 0$ in Eq. (34), we get
\[ w_{2(2)}(x, t) = \left\{ \lambda_0 - \lambda_0 \left[ \tanh(2\gamma \sqrt{-\frac{\lambda_0}{\lambda_2}}) \pm i \text{sech}(2\gamma \sqrt{-\frac{\lambda_0}{\lambda_2}}) \right] \right\}^{\frac{n}{2}}. \tag{36} \]

Setting $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{-\frac{k}{\gamma}} = \pm 2\sqrt{-\frac{2m}{\lambda_2}}$, and $\frac{4}{\gamma} = \frac{2m}{\lambda_2} > 0$ in Eq. (34), we have
\[ w_{2(3)}(x, t) = \left\{ \lambda_0 + \lambda_0 \left[ \tan(2\gamma \sqrt{-\frac{\lambda_0}{\lambda_2}}) \mp \sec(2\gamma \sqrt{-\frac{\lambda_0}{\lambda_2}}) \right] \right\}^{\frac{n}{2}}. \tag{37} \]

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{k}{\gamma}} = \pm 2\sqrt{-\frac{2m}{\lambda_2}}$, and $\frac{4}{\gamma} = \frac{2m}{\lambda_2} > 0$ in Eq. (34), we have
\[ w_{2(4)}(x, t) = \left[ \frac{2\lambda_0}{1 \pm \cos(2\gamma \sqrt{\frac{2m}{\lambda_2}})} \right]^\frac{1}{n}. \tag{38} \]
Family 3

\[ w_3(x,t) = \left\{ \mu_0 + \mu_2 \left[ \frac{(\gamma a_0^2 + Ab_0^2)}{4Ab_{-1}} \exp(2\eta) + b_0 + b_{-1} \exp(-2\eta)} \right]^2 \right\}^{\frac{1}{n}}, \]

where \( \xi = k \left[ x + \frac{4k^2 \mu_0}{n^2 \mu_0} \int g(t) dt - C_2 \right] \), \( f(t) = -\frac{2(n+1)(n+2)g(t)k^2 \gamma^2 \mu_2}{n^2 \mu_0} \), \( A = \frac{\mu_2 \gamma}{\mu_0} \), \( \eta = \gamma \sqrt{-\frac{4}{\gamma} \xi} \).

If we set \( b_0 = 0, b_{-1} = i, a_0 = \pm 2 \sqrt{-\frac{A}{\gamma}} = \pm 2 \sqrt{-\frac{\mu_2}{\mu_0}}, \) and \( A = \frac{\mu_2}{\mu_0} < 0 \) in Eq. (39), we obtain

\[ w_{3(1)}(x,t) = \left\{ \mu_0 - \frac{\mu_0}{\left[ \tanh(2\gamma \sqrt{-\frac{\mu_2}{\mu_0}} \xi) \pm \text{sech}(2\gamma \sqrt{-\frac{\mu_2}{\mu_0}} \xi) \right]^2} \right\}^{\frac{1}{n}}. \]  \( \text{(40)} \)

Setting \( b_0 = 0, b_{-1} = 1, a_0 = \pm 2 \sqrt{\frac{A}{\gamma}} = \pm 2 \sqrt{\frac{\mu_2}{\mu_0}}, \) and \( A = \frac{\mu_2}{\mu_0} > 0 \) in Eq. (39), we get

\[ w_{3(2)}(x,t) = \left\{ \mu_0 + \frac{\mu_0}{\left[ \tan(2\gamma \sqrt{\frac{\mu_2}{\mu_0}} \xi) \pm \sec(2\gamma \sqrt{\frac{\mu_2}{\mu_0}} \xi) \right]^2} \right\}^{\frac{1}{n}}. \]  \( \text{(41)} \)

4 The exact solutions of Eq. (2)

In this section, we proceed in a way analogous to the case of Eq. (1). In order to obtain new exact travelling wave solutions for Eq. (2), we use

\[ u(x,t) = u(\xi), \quad \xi = B(x - wt), \]  \( \text{(42)} \)

where \( B \) and \( w \) are constants, and substituting the (42) into Eq. (2), we obtain

\[ -w(u')' + au^m u' + bB^2(u^n)' = 0. \]  \( \text{(43)} \)

Balancing the order of the nonlinear term \( u^m u' \) with the term \( (u^n)' \) in (43), we obtain

\[ mP + P + 1 = nP + 3, \]

so that

\[ P = \frac{2}{m - n + 1}. \]  \( \text{(44)} \)
To get a closed form solution, it is natural to use the transformation
\[ u = v_{m-n+1}, \] (45)
and when \( l = n \), Eq. (43) becomes
\[ -wl(m-n+1)^2v^2v' + a(m-n+1)^2v'v^3 + bB^2[n(2n-m-1)(3n-2m-2) \times (v')^3 + 3n(2n-m-1)(m-n+1)vv'' + n(m-n+1)^2v^2v'']. \] (46)

This means that all the evolution terms that satisfy the condition \( l = n \) contribute to the soliton formation.

Now, we assume that the solution of Eq. (46) can be expressed in the following form
\[ v = v(\xi) = \sum_{j=0}^{N} \alpha_j \phi^j(\xi) + \sum_{j=1}^{N} \beta_j \phi^{-j}(\xi), \] (47)
where \( N \) is positive integers which are given by the homogeneous balance principle, \( \phi(\xi) \) is a solution of Eq. (4). Balancing \( v'v^3 \) term with \( v^2v''' \) term in (46) gives \( N = 2 \). Therefore, we obtain
\[ v = \alpha_0 + \alpha_1 \phi(\xi) + \alpha_2 \phi^2(\xi) + \frac{\beta_1}{\phi(\xi)} + \frac{\beta_2}{\phi^2(\xi)}. \] (48)

Substituting Eq. (48) into (46) and using the Riccati equation (4), collecting the coefficients of \( \phi(\xi) \), we have
\[ \frac{1}{D} \left[ C_0 + C_1 \phi(\xi) + C_2 \phi^2(\xi) + \cdots + C_{15} \phi^{15}(\xi) + C_{16} \phi^{16}(\xi) \right] = 0. \] (49)

Because the expresses to these coefficients \( D, C_0 = 0, C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0, \cdots, C_{15} = 0, C_{16} \) of \( \phi(\xi) \) in Eq. (49) are too lengthiness, so we omit them. But we can directly use the command "solve" in mathematical software Maple to solve the following set of algebraic equations
\[ C_0 = 0, \ C_1 = 0, \ C_2 = 0, \ C_3 = 0, \ C_4 = 0, \cdots, \ C_{15} = 0, \ C_{16} = 0. \] (50)
Solved the above algebraic equations, we obtain the following three sets of solutions

**Case 1**
\[ \omega = -\frac{4br\gamma AB^2n^2}{(m-n+1)^2}, \quad \alpha_0 = -\frac{2nbA\gamma B^2(m+n+1)(m+1)}{a(m-n+1)^2}, \quad \alpha_1 = 0, \]
\[ \alpha_2 = -\frac{2nb\gamma B^2(m+n+1)(m+1)}{a(m-n+1)^2}, \quad \beta_1 = 0, \quad \beta_2 = 0. \] (51)
Case 2
\[\omega = -\frac{4b\gamma AB^2 n^2}{(m - n + 1)^2}, \quad \alpha_0 = -\frac{2nbA\gamma B^2(m + n + 1)(m + 1)}{a(m - n + 1)^2}, \quad \alpha_1 = 0, \]
\[\alpha_2 = 0, \quad \beta_1 = 0, \quad \beta_2 = -\frac{2nbA^2 B^2(m + n + 1)(m + 1)}{a(m - n + 1)^2}. \quad (52)\]

Case 3
\[\omega = -\frac{16b\gamma AB^2 n^2}{(m - n + 1)^2}, \quad \alpha_0 = -\frac{4nbA\gamma B^2(m + n + 1)(m + 1)}{a(m - n + 1)^2}, \quad \alpha_1 = 0, \]
\[\alpha_2 = -\frac{2nb\gamma^2 B^2(m + n + 1)(m + 1)}{a(m - n + 1)^2}, \quad \beta_1 = 0, \]
\[\beta_2 = -\frac{2nbA^2 B^2(m + n + 1)(m + 1)}{a(m - n + 1)^2}. \quad (53)\]

Thus from Eqs. (48), (51), (52), and (53), we obtain families of exact solutions to Eq. (46) as follows,

\[
v = -\frac{2nbB^2(m + n + 1)(m + 1)}{a(m - n + 1)^2} \left[ A\gamma + \gamma^2 \phi^2(\xi) \right], \quad (54)\]
\[
v = -\frac{2nbB^2(m + n + 1)(m + 1)}{a(m - n + 1)^2} \left[ A\gamma + A^2 \frac{1}{\phi^2(\xi)} \right], \quad (55)\]
\[
v = -\frac{2nbB^2(m + n + 1)(m + 1)}{a(m - n + 1)^2} \left[ 2A\gamma + \gamma^2 \phi^2(\xi) + A^2 \frac{1}{\phi^2(\xi)} \right], \quad (56)\]

where \(\phi(\xi)\) is a solution of the Riccati Eq. (4).

Substituting solutions (11)-(18) of the Riccati Eq. (4) into (54), (55) and (56), using the transformation (45), we have the following several families of solutions to Eq. (2).

Family 1

\[
u_1(x, t) = \left( RA\gamma + R\gamma^2 \left[ -\frac{a}{2} b_1 \exp(\gamma \sqrt{-A\xi}) + a_{-1} \exp(-\gamma \sqrt{-A\xi}) b_1 \exp(\gamma \sqrt{-A\xi}) + a_{-1} \exp(-\gamma \sqrt{-A\xi}) \right] \right)^{\frac{1}{m-n+1}} \frac{1}{m-n+1}, \quad (57)\]

where \(R = -\frac{2nbB^2(m+n+1)(m+1)}{a(m-n+1)^2}\), \(\xi = B \left[ x + \frac{4b\gamma AB^2 n^2 l}{(m-n+1)^2} \right], \quad l = n.\)
If we set $b_1 = 1$, $a_{-1} = \pm \sqrt{-\frac{4}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in Eq. (57), we obtain

$$u_{1(1)}(x, t) = \left( -\frac{2nA\gamma b^2(m + n + 1)(m + 1)}{a(m - n + 1)^2} \sech^2(\gamma \sqrt{\frac{A}{\gamma}} a_{-1}) \right) \frac{1}{m-n+1}, \quad (58)$$

and

$$u_{1(2)}(x, t) = \left( \frac{2nA\gamma b^2(m + n + 1)(m + 1)}{a(m - n + 1)^2} \csch^2(\gamma \sqrt{-\frac{A}{\gamma}} a_{-1}) \right) \frac{1}{m-n+1}. \quad (59)$$

Setting $b_1 = i$, $a_{-1} = \mp \sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in Eq. (57), we get

$$u_{1(3)}(x, t) = \left( \frac{2nA\gamma b^2(m + n + 1)(m + 1)}{a(m - n + 1)^2} \sec^2(\gamma \sqrt{\frac{A}{\gamma}} a_{-1}) \right) \frac{1}{m-n+1}, \quad (60)$$

and

$$u_{1(4)}(x, t) = \left( \frac{2nA\gamma b^2(m + n + 1)(m + 1)}{a(m - n + 1)^2} \csc^2(\gamma \sqrt{\frac{A}{\gamma}} a_{-1}) \right) \frac{1}{m-n+1}. \quad (61)$$

**Family 2**

$$u_2(x, t) = \left( 2RA\gamma + R\gamma^2 \left[ \frac{-b_1 \exp(\gamma \sqrt{-\frac{A}{\gamma}} a_{-1} \exp(-\gamma \sqrt{-\frac{A}{\gamma}} a_{-1}))}{b_1 \exp(\gamma \sqrt{-\frac{A}{\gamma}} a_{-1} \exp(-\gamma \sqrt{-\frac{A}{\gamma}} a_{-1}))} \right]^2 \right) \frac{1}{m-n+1}, \quad (62) \right.$$

where $R = -\frac{2nbb^2(m+n+1)(m+1)}{a(m-n+1)^2}$, $\xi = B \left[ x + \frac{16b\gamma AB^2n^2t}{(m-n+1)^2} \right]$, $l = n$.

If we set $b_1 = 1$, $a_{-1} = \pm \sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in Eq. (62), we obtain

$$u_{2(1)}(x, t) = \left( RA\gamma \left[ 2 - \tanh^2(\gamma \sqrt{-\frac{A}{\gamma}} a_{-1}) \right] - \coth^2(\gamma \sqrt{-\frac{A}{\gamma}} a_{-1}) \right) \frac{1}{m-n+1}. \quad (63)$$

Setting $b_1 = i$, $a_{-1} = \mp \sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in Eq. (62), we get

$$u_{2(2)}(x, t) = \left( RA\gamma \left[ 2 + \tan^2(\gamma \sqrt{\frac{A}{\gamma}} a_{-1}) + \cot^2(\gamma \sqrt{\frac{A}{\gamma}} a_{-1}) \right] \right) \frac{1}{m-n+1}. \quad (64)$$
Family 3

\[ u_3(x, t) = \left( RA\gamma + R\gamma^2 \left[ \frac{(\gamma a_n^2 + Ab^2_n)}{4\gamma\sqrt{-\frac{A}{\gamma}}} \exp(2\eta) + a_0 + \sqrt{-\frac{A}{\gamma}} b_{-1} \exp(-2\eta) \right] \right)^{\frac{1}{m-n+1}}, \]

where \( R = -\frac{2nbB^2(m+n+1)(m+1)}{a(m-n+1)^2}, \quad \xi = B \left[ x + \frac{4b\gamma AB^2n^2}{(m-n+1)^2} t \right], \quad l = n, \quad \eta = \gamma\sqrt{-\frac{A}{\gamma}}\xi. \)

If we set \( b_0 = 0, b_{-1} = 1, a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}, \) and \( \frac{A}{\gamma} < 0 \) in Eq. (65), we obtain

\[ u_{3(1)}(x, t) = \left( -\frac{4n\gamma bB^2(m+n+1)(m+1)}{a(m-n+1)^2 \left[ 1 \mp \cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right]} \right)^{\frac{1}{m-n+1}}. \]  

(66)

Setting \( b_0 = 0, b_{-1} = i, a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}, \) and \( \frac{A}{\gamma} < 0 \) in Eq. (65), we get

\[ u_{3(2)}(x, t) = \left\{ RA\gamma \left( 1 - \left[ \tanh(2\gamma\sqrt{-\frac{A}{\gamma}}) \pm i \text{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}) \right] \right) \right\}^{\frac{1}{m-n+1}}. \]

(67)

Setting \( b_0 = 0, b_{-1} = 1, a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}, \) and \( \frac{A}{\gamma} > 0 \) in Eq. (65), we have

\[ u_{3(3)}(x, t) = \left\{ RA\gamma \left( 1 + \left[ \tan(2\gamma\sqrt{\frac{A}{\gamma}}) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}) \right] \right) \right\}^{\frac{1}{m-n+1}}. \]

(68)

Setting \( b_0 = 0, b_{-1} = i, a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}, \) and \( \frac{A}{\gamma} > 0 \) in Eq. (65), we have

\[ u_{3(4)}(x, t) = \left( -\frac{4n\gamma bB^2(m+n+1)(m+1)}{a(m-n+1)^2 \left[ 1 \pm \cos(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \right]} \right)^{\frac{1}{m-n+1}}. \]

(69)

Family 4

\[ u_4(x, t) = \left( RA\gamma + RA^2 \left[ \frac{(\gamma a_n^2 + Ab^2_n)}{4\gamma\sqrt{-\frac{A}{\gamma}}\exp(2\eta) + b_{0} + b_{-1} \exp(-2\eta) \right] \right)^{\frac{1}{m-n+1}}, \]

\[ \left[ \frac{(\gamma a_n^2 + Ab^2_n)}{4\gamma\sqrt{-\frac{A}{\gamma}}\exp(2\eta) + a_0 + \sqrt{-\frac{A}{\gamma}} b_{-1} \exp(-2\eta) \right] \right)^2. \]

(70)
where \( R = -\frac{2nbB^2(m+n+1)(m+1)}{a(m-n+1)^2} \), \( \eta = \gamma \sqrt{-\frac{4}{\gamma}} \xi \).

If we set \( b_0 = 0, \ b_{-1} = i, \ a_0 = \pm 2 \sqrt{-\frac{4}{\gamma}}, \) and \( \frac{4}{\gamma} < 0 \) in Eq. (70), we obtain

\[
\eta = \gamma \sqrt{-\frac{4}{\gamma}} \xi, \quad \xi = B \left[ x + \frac{4b\gamma AB^2n^2}{(m-n+1)^2} t \right], \quad l = n,
\]

Setting \( b_0 = 0, \ b_{-1} = 1, \ a_0 = \pm 2 \sqrt{\frac{4}{\gamma}}, \) and \( \frac{4}{\gamma} > 0 \) in Eq. (70), we get

\[
u_{4(2)}(x, t) = \left\{ RA\gamma \left( 1 - \frac{1}{\left[ \tanh(2\gamma \sqrt{-\frac{4}{\gamma}} \xi) \pm \text{sech}(2\gamma \sqrt{-\frac{4}{\gamma}} \xi) \right]^2} \right) \right\}^{\frac{1}{m-n+1}}. \tag{72}
\]

**Family 5**

\[
u_5(x, t) = \left( 2RA\gamma + R\gamma^2 \left[ \frac{(\gamma n^2 + 4b_0^2)}{4b_{-1}} \exp(2\eta) + a_0 + \sqrt{-\frac{4}{\gamma}} b_{-1} \exp(-2\eta) \right]^2 + RA^2 \left[ \frac{(\gamma n^2 + 4b_0^2)}{4b_{-1}} \exp(2\eta) + a_0 + \sqrt{-\frac{4}{\gamma}} b_{-1} \exp(-2\eta) \right] \right)^{\frac{1}{m-n+1}}, \tag{73}
\]

where \( R = -\frac{2nbB^2(m+n+1)(m+1)}{a(m-n+1)^2} \), \( \eta = \gamma \sqrt{-\frac{4}{\gamma}} \xi \), \( \xi = B \left[ x + \frac{16b\gamma AB^2n^2}{(m-n+1)^2} t \right], \)

If we set \( b_0 = 0, \ b_{-1} = 1, \ a_0 = \pm 2 \sqrt{-\frac{4}{\gamma}}, \) and \( \frac{4}{\gamma} < 0 \) in Eq. (73), we obtain

\[
u_{5(1)}(x, t) = \left( \frac{8nA\gamma bB^2(m + n + 1)(m + 1)}{a(m-n+1)^2 \left[ \cosh^2(2\gamma \sqrt{-\frac{4}{\gamma}} \xi) - 1 \right]} \right)^{\frac{1}{m-n+1}}. \tag{74}
\]

Setting \( b_0 = 0, \ b_{-1} = 1, \ a_0 = \pm 2 \sqrt{\frac{4}{\gamma}}, \) and \( \frac{4}{\gamma} > 0 \) in Eq. (73), we have

\[
u_{5(2)}(x, t) = \left( -\frac{8nA\gamma bB^2(m + n + 1)(m + 1)}{a(m-n+1)^2} \sec^2(2\gamma \sqrt{\frac{A}{\gamma}} \xi) \right)^{\frac{1}{m-n+1}}. \tag{75}
\]

Setting \( b_0 = 0, \ b_{-1} = i, \ a_0 = \pm 2 \sqrt{\frac{4}{\gamma}}, \) and \( \frac{4}{\gamma} > 0 \) in Eq. (73), we have
\[ u_{5(3)}(x, t) = \left( \frac{8nA\gamma bB^2(m + n + 1)(m + 1)}{a(m - n + 1)^2 \left[ \cos^2(2\gamma \sqrt{\frac{\xi}{\gamma}}) - 1 \right]} \right)^{\frac{1}{m-n+1}}. \quad (76) \]

5 Conclusions

In this work we studied the generalized KdV equation with time-dependent coefficients. In addition, the \( K(m, n) \) equation with generalized evolution is also examined. The Riccati equation mapping method is used to carry out this work. Some new exact solutions including the solitary wave solutions and the periodic wave solutions are obtained for both equations. It is worth while to mention that this method is reliable and effective in solving nonlinear partial differential equations. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations.

References


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