Maximum Principle and Error Estimate of the Upwind Finite Volumes Method for Nonlinear Convection-Diffusion Equation

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Abstract. In this paper we used the combined F.E.-F.V. method to solve an initial-boundary value problem, provided that the nonlinear convective term is approximated by U.F.V. method on the finite volume mesh dual to a triangular grid of weakly acute type and the diffusion term is approximated by standard Galerkin finite element method. We proved the stability and the discrete maximum principle under the condition \( 0 \leq \tau \leq \left| \Omega \right| c(M) \left| \partial \Omega \right|, \ i \in J \). Also proved the approximate solution is convergent with error of \( O(\tau) \) and the local truncation with error of \( O(\tau^2) \).

Keywords: finite volumes, Maximum Principle, Error estimate, Convection-Diffusion problem

1. Introduction

Convection-diffusion processes appear in many areas of science and technology, and environmental protection. These problems are peculiar in the sense that it is a combination of
two dissimilar phenomena, convection and diffusion. It is a well-known fact that the use of classical Galerkin method with continuous piecewise linear finite elements leads to spurious oscillations when the local Peclet number is large. This is the reason that the numerical solution of convection-diffusion problems attracts a number of specialists. From an extensive literature devoted to problems let us mention the papers [1], [6], [12], [11], [10], [3], and [5]. In the theory of weak solutions for partial differential equations in divergence from there are two roughly equivalent formulations in common use, namely, the functional formulation involving integration against smooth test functions versus the finite volume type over arbitrary control volumes. The former corresponds to energy methods and leads naturally to F.E discretizations for elliptic and parabolic, i.e., diffusive, problems. The latter corresponds in a natural way to the physical formulation of the basic laws of mass, momentum, and energy in fluid mechanics leading directly to the well-known FV methods. In [9] the work reported here we investigate the approach by trying to have the best of both worlds, i.e., the combination of finite volumes for in viscid conservation laws with finite elements for the diffusion. Computational fluid dynamics problems usually require discretization of the problem into a large number of cells/grid points (millions and more), therefore cost of the solution favors simpler, lower order approximation within each cell/grid point. This is especially true of "external flow" problems, like air flow around the car or airplane, or weather simulation in a large area. We view space as being broken down into a set of volumes each of which surrounds one point. In particular the volume associated with each point is the region of space that is closer to that point than to any other point. Now we consider how can be used these cells to solve fluid flow problem. The way to think about it is to consider the total flow that enters a cell through its boundaries. It is claimed that the net flow (sum of flow in and out) has to be zero. Intuitively this is because the fluid (or air) has nowhere to go. This condition can be written formally as

$$\int_{\partial \Omega_i} \bar{u} \cdot \bar{n} \, ds = 0$$

where $\partial \Omega_i$ is the boundary of the $i^{th}$ cell, $\bar{u}$ is the flow and $\bar{n}$ is the vector normal to the surface let us mention the papers [11], [10], and [9]. Our main goal is to develop a robust theoretically based numerical method for the solution of viscous compressible flow applied of unstructured meshes. In this paper we proposed numerical schemes for the solution of viscous gas flow based on the combination of the FV method for the discretization of inviscid convective terms and the FE method applied to the approximation of viscous terms.

This paper consists

2. The Convection-Diffusion Equation

We consider the two–dimensional nonlinear convection–diffusion equation [6].
Maximum principle and error estimate

\[ \frac{\partial u}{\partial t} - \lambda \Delta u + \nabla \cdot (\vec{b}(u) \cdot u) = f(x,t) \quad \Omega \times (0,T) = Q_T, \]  

\[ u(x, t) = 0, \quad \text{on } \partial \Omega, \]  

\[ u(x, 0) = u^o(x), \quad \text{on } \Omega \]  

where \( x = (x_1, x_2) \) and \( \lambda > 0 \) is a given constant, the vector \( \vec{b}(u) : Q_T \rightarrow \mathbb{R}^2 \) is a convection coefficient, the functions \( f(x,t) : Q_T \rightarrow \mathbb{R} \) \( (f = f(x,t) \text{ for simplicity}) \) and the function \( u^o : \Omega \rightarrow \mathbb{R} \) are given. The weak form is to find \( u : [0,T] \rightarrow H^1_0(\Omega) \), for simplicity we will denote \( u = u(t) = u(.,t) : \)

\[ \left( \frac{d}{dt} u(t), v \right) + (\lambda \nabla u(t), \nabla v) + (\nabla \cdot (\vec{b}(u(t)) \cdot u(t)), v) = (f,v), \quad \forall v \in H^1_0(\Omega), \]

\[ u(0) = u^0. \]  

Assumption 2.1. [11]

a) \( f \in C( [0,T], W^{1,q}(\Omega) ) \) for some \( q > 2 \).

b) \( u^o \in W^{1,p}(\Omega) \) for some \( p > 2 \).

c) \( \) a positive constants \( \gamma_1 \) and \( \gamma_2 \) such that

\[ \| u^0 \|_{L^\infty(\Omega)} \leq \gamma_1, \quad \| f \|_{L^\infty(Q_T)} \leq \gamma_2. \]

Now, we define the set \( Y = \left\{ u : \| u \| \leq M \right\} \) and the coefficients of equation (2.1) satisfy the following conditions

**A (2.1)** \( \vec{b}(u) = (b_1(u), b_2(u)) \in W^{1,\infty}(Y) \times W^{1,\infty}(Y). \)

**A (2.2)** \( \vec{b}(u) \) is locally Lipschitz –continuous \( \| \vec{b}(u) - \vec{b}(v) \| \leq L \| u - v \|, \quad \forall u, v \in Y \)

Where \( 0 < L < 1 \) is Lipchitz constant?

**A (2.3)** \( u \in L^\infty(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \), \( u' \in L^\infty(0,T;L^2(\Omega)) \) and \( u'' \in L^\infty(0,T;V^*), \)

where \( u \) is the weak solution of equation (2.4) and \( V^* \) is the dual space of \( V \).

**Lemma 2.1.** [5] Let \( U^k \) and \( u^k \) are the approximate solution and the exact solution

Respectively, if \( \| U^k - u^k \|_{L^2(\Omega)} \leq Ch \), \( t_k \in (0,T) \) then

\[ \| \nabla U^k \|_{L^2(\Omega)} \leq C_1, \]

where \( C_i \) is a constant independent of \( h \) and \( \tau \).

**Lemma 2.2.** [8] Let \( \varphi \in H^2(\Omega) \cap V \) and \( \varphi_h = p_i\varphi \) such that \( \varphi \)}
\[ \| p_h \varphi - \varphi \|_{H^1_0(\Omega)} \leq c h \| \varphi \|_{H^2(\Omega)}, \quad h \in (0, h_0), \]
where \( p_h \) is the projection and the constant \( c \) is independent of \( \varphi \) and \( h \).

**Lemma 2.3.** [10] Let \( \varphi \in H^2(\Omega) \cap V \) and \( \varphi_h = p_h \varphi \) such that
\[ \| p_h \varphi - \varphi \|_{L^2(\Omega)} \leq c h^2 \| \varphi \|_{H^2(\Omega)}, \quad h \in (0, h_0), \]
where \( p_h \) is the projection and the constant \( c \) is independent of \( \varphi \) and \( h \).


The F.E. method, in its simplest form, is a specific process of constructing subspaces \( V_h \subset V \), which are called F.E. spaces. The most characteristic is discretization such as triangulation \( \mathcal{T}_h \) established over the closure \( \tilde{\Omega} = \Omega \cup \partial \Omega \) of \( \Omega \subset \mathbb{R}^2 \). Let \( \Omega_h \) be a polygonal approximation of the domain \( \Omega \), thus denotes a partition of \( \Omega \) into disjoint triangles \( e \) such that the following properties are satisfied

\begin{enumerate}
  \item \( \tilde{\Omega}_h = \bigcup_{e \in \mathcal{T}_h} e \),
  \item If \( e_1, e_2 \in \mathcal{T}_h \), then \( e_1^o \cap e_2^o = \emptyset \) or \( e_1 \) and \( e_2 \) have a common side or \( e_1 \) and \( e_2 \) have a common vertex,
  \item \( P \in \tilde{\Omega}_h \) for any vertex \( P \) of each \( e \in \mathcal{T}_h \).
\end{enumerate}

Then the triangulation \( \mathcal{T}_h \) is called a basic mesh.

**Remark 3.1[11].** We associate the index set \( J = \{i \neq j; p_i \in \tilde{\Omega}_h\} \) and \( \hat{J} = \{i \in J; p_i \in \Omega_h\} \) be the set of the indices of all interior vertices. Let \( \mathcal{P}_h = \{p_i; i \in J\} \) be the set of all vertices of all \( e \in \mathcal{T}_h \). Let \( \phi_i, \ i \in J \) be the continuous function in \( \tilde{\Omega}_h \) such that \( \phi_i(p) \) is a linear on each \( e \in \mathcal{T}_h \) and \( \phi_i(p_j) = \delta_{ij} \) for all \( p_j, i, j \in J \), where \( \delta_{ij} \) is Kronecker’s delta. The linear F.E. spaces \( V_h \) and \( V_{0h} \)
\[ V_h = \{ \phi : \phi \in C(\tilde{\Omega}_h), \phi \text{ is linear function for each } e \in \mathcal{T}_h \}, \]
\[ V_{0h} = \{ \phi : \phi \in V_h, \phi = 0 \text{ on } \partial \Omega_h \}. \]
By $I_h$ we denote the operator of the interpolation $I_h : C(\Omega_h) \to V_{0h}$. Hence, if $v : \mathcal{T}_h \to \mathbb{R}$, then 

$$I_h v \in V_{0h}, \quad (I_h v)(p_i) = v(p_i), \quad p_i \in \mathcal{T}_h.$$ 

### 4. The Dual Finite Volumes

To construct the dual mesh $\hat{\Omega}_h = \{ \hat{\Omega}_i ; i \in J \}$ over the basic mesh $\mathcal{T}_h$, the dual F.V. $\hat{\Omega}_i$ associated with a vertex $P_i \in \mathcal{T}_h$ is a closed polygon obtained in the following way [11]: By joining the center of every triangle $e \in \mathcal{T}_h$ that contains the vertex $P_i$ with the center of every side of $e$ containing $P_i$. If $P_i \in \mathcal{T}_h \cap \partial \Omega$, then complete the obtained contour by the straight segments joining $P_i$ with the centers of boundary sides (i.e. sides which are subsets of $\partial \Omega$) that contain $P_i$. In this way, one can get the boundary $\partial \hat{\Omega}_i$ of the finite volume $\hat{\Omega}_i$ (see Figure 4.1). It is obvious that 

$$\tilde{\Omega}_h = \bigcup_{i \in J} \hat{\Omega}_i.$$ 

The interiors of $\hat{\Omega}_i$, $i \in J$, are mutually disjoint. If for two different F.V. $\hat{\Omega}_i$ and $\hat{\Omega}_j$ their boundaries contain a common straight segment, which call neighbors, then 

$$\Gamma_y = \bigcup_{a=1}^{\gamma_y} \partial \hat{\Omega}_i \cap \partial \hat{\Omega}_j = \Gamma_y,$$

where the set $\gamma_y$ consists of one or two straight segments $\gamma_y$ and $\Gamma_y = \Gamma_y$ as shown in Figure 4.1. We have 

$$\ell_y = \begin{cases} 2 & \text{for } \hat{\Omega}_i \text{ or } \hat{\Omega}_j \subset \hat{\Omega}_h, \\ 1 & \text{for } \hat{\Omega}_i \text{ and } \hat{\Omega}_j \text{ are adjacent to } \partial \hat{\Omega}_h. \end{cases}$$

If $P_i \in \mathcal{T}_h \cap \partial \hat{\Omega}_h$, then we denote by $\Gamma_{i-1}^{\alpha} = \ell_{i-1}$ for $\alpha = 1, 2$, the segments that form $\partial \hat{\Omega}_i \cap \partial \hat{\Omega}_h$. Put 

$$S(i) = \begin{cases} s(i) \cup \{-1\} & \text{for } P_i \in \mathcal{T}_h \cap \partial \hat{\Omega}_h, \\ s(i) & \text{for } P_i \in \mathcal{T}_h \cap \hat{\Omega}_h. \end{cases}$$
Figure 4.1: Basic triangular mesh in a domain $\hat{\Omega}_h$ and the dual F.V.

For every $\hat{\Omega}_i \in \hat{\Omega}_h$ we have

$$\partial \hat{\Omega}_i = \bigcup_{j \in S(i)} \Gamma_{ij} = \bigcup_{j \in S(i)} \bigcup_{a=1}^{\hat{\alpha}_j} \Gamma_{ij}^a.$$  

(4.1)

the open segments obtained by removing the endpoints from $\Gamma_{ij}^a$ are mutually disjoint.

Remark 4.1. We introduce the following notations

a) $|e|$ = area of $e \in T_h$.

b) $\beta_{ij} = |\Gamma_{ij}| = \text{length of } \Gamma_{ij}$, $\beta_{ij}^a = |\Gamma_{ij}^a|$.

c) $|\hat{\Omega}_i|$ = area of $\hat{\Omega}_i \in \hat{\Omega}_h$, $|\partial \hat{\Omega}_i|$ = length of $\partial \hat{\Omega}_i$.

d) $n_{ij}^a = (n_{ij}^a_1, n_{ij}^a_2)$ = unit outer normal to $\partial \hat{\Omega}_i$ on $\Gamma_{ij}^a$.

e) Let us consider a partition $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = T$ of the time interval $(0,T)$ and set $\tau = t_{k+1} - t_k$ for $k = 0,1,\ldots$.


a) Triangulation family $T_h$ is of weakly acute type. This means all angles of all $e \in T_h$, are less than or equal to $\pi/2$.

b) Assume that the domain $\Omega$ is polygonal and thus $\hat{\Omega}_h = \hat{\Omega}$.

c) Define $h_e = \text{the length of the longest side of the triangle } e \in T_h$.
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\[ \theta_e = \text{the magnitude of the smallest angle of the triangle } e \in \mathcal{T}_h, \]

and put \( h = \max_{e \in \mathcal{T}_h} h_e, \quad \theta_h = \min_{e \in \mathcal{T}_h} \theta_e. \)

d) Triangulation family \( \mathcal{T}_h, \ h \in (0, h_0), \) be regular, i.e. there exists \( \delta_0 > 0 \)

such that, \( \theta_h \geq \delta_0 > 0 \quad \forall h \in (0, h_0). \)

e) In view of Assumption (c) which implies the existence of a constant \( \hat{c} > 0 \) such that

\[ h^2 \leq \hat{c}|e|, \quad e \in \mathcal{T}_h, \ h \in (0, h_0). \]

f) Introduce the inverse stability assumption \( h = 0(\tau) \) it means that we consider the condition

\[ h \leq \bar{c}\tau \]

where a constant \( \bar{c} > 0 \) independent of \( h \) and \( \tau. \)

5. The Upwind Finite Volumes Scheme

Let \( \tau > 0 \) be the time step, the U.F.V. scheme for \( t_k \in [0, T) \) to find \( U^{k+1} \in V_{0h} \) such that

\[
\left( \frac{U^{k+1} - U^k}{\tau}, v_h \right)_h + (\lambda \nabla U^{k+1}, \nabla v_h)_h + R(\vec{B}^k, U^k, v_h) = (f^{k+1}, v_h)_h, \quad \forall v_h \in V_{0h},
\]

\[ U^0 = I_h u^0, \quad (5.1) \]

Where \( I_h \) is the interpolation operator and \( R(\vec{B}^k, U^k, v) \) is derived as follows: By using (4.1)

and the definition of the mass lumping operator \( L_h \) for all \( u, v \in V_h \) we get \( [13]: \)

\[
(\nabla(\vec{b}(u), v)) = \int_{\Omega} (\nabla(\vec{b}(u))) v dx \approx \sum_{i \in J} (\nabla(\vec{b}(u))) v dx \\
= \sum_{i \in J} v(p_i) \int_{\Omega_i} (\nabla(\vec{b}(u))) v dx = \sum_{i \in J} v(p_i) \int_{\Omega_i} (\vec{b}(u)) n u ds \\
= \sum_{i \in J} v(p_i) \sum_{j \in S(i)} \sum_{a=1}^{l_{ij}} \int_{\Gamma_{ij}} (\vec{b}(u)) n u ds, \quad (5.2)
\]

replacing the function \( u \) on \( \Gamma_{ij}^\alpha \) by some convex combination of the nodal values of \( u_i \) and \( u_j \) with parameter \( H_{ij}^\alpha \)

\[
(\nabla(\vec{b}(u), v)) \approx \sum_{i \in J} v(p_i) \sum_{j \in S(i)} \sum_{a=1}^{l_{ij}} (H_{ij}^\alpha u_i + H_{ji}^\alpha u_j) \int_{\Gamma_{ij}^\alpha} (\vec{b}(u)) n u ds,
\]

then we define
\[ R(\vec{B}^k, U^k, v) = \sum_{i=1}^{\Omega} \sum_{j=1}^{n} \int_{\Omega} (H^k_j)^{-1} U^k_j + H^k_j U^k_j) \beta^{k,\alpha}_{ij}, \]  

(5.3)

where \( U^k_j = U^k(p_j), v_i = v(p_i) \) and

\[ \beta^{k,\alpha}_{ij} = \int_{\Gamma} \vec{B}^k n^\alpha ds, \quad \vec{B}^k = b(U^k). \]

Where the function \( H: \mathbb{R}^2 \times S \rightarrow \mathbb{R}, \quad S = \{ n \in \mathbb{R}^2, |n| = 1 \} \) is called a numerical flux.

With \( \phi_i, \phi_j \) the basis functions in \( H^0(\Omega) \).

5.1. The Properties of the Numerical Flux ([11], [7]) We use the following assumptions

a) \( H = H(y, z, n) \) is locally Lipschitz-continuous with respect to \( y, z \)
   
   for any \( M \geq 0 \) there exists a constant \( c(M) > 0 \) such that
   
   \[ |H(y, z, n) - H(y^*, z^*, n)| \leq c(M)(|y - y^*| + |z - z^*|) \quad \forall y, y^*, z, z^* \in [-M, M]. \]

b) \( H \) is consistent
   
   \[ H(u, u, n) = b(u)\nu, \quad \forall u \in V. \]

c) \( H \) is conservative
   
   \[ H(y, z, n) = -H(z, y, -n) \quad \forall y, z \in V. \]

d) \( H \) is monotone in the following sense: For a given fixed number \( M > 0 \), the function \( H(y, z, n) \) is non-increasing with respect to the second variable \( z \) on the set \( \zeta_M = \{(y, z, n); y, z \in [-M, M]\} \).


a) If \( i \in J \) and \( e \in T_h \) is a triangle with the vertex \( p_i \in P_h \) then
   
   \[ |e \cap \Omega| = \frac{1}{3} |e|. \]

b) The approximation \( (\cdot, \cdot)_h \) of the \( L^2 \)-scalar product can be defined with the aid of numerical integration using the vertices \( P^e_1, P^e_2, P^e_3 \) of \( e \in T_h \), as the integration points

\[ (u, v)_h = \sum_{e \in T_h} \int_{\Omega} (\phi^e_1 u(P^e_1) v(P^e_1))^2 = \frac{1}{3} (L_h u)(L_h v), \quad u, v \in C(\Omega_h), \]

\[ \|u\|_{L^2(\Omega)} = \|L_h u\|_{L^2(\Omega)}, \quad u \in C(\Omega_h). \]
c) We have

\[
(\phi_i, \phi_i)_h = \begin{cases} 
\hat{\Omega}_i, & i = j \\
0, & i \neq j 
\end{cases}
\]

\[
(u, \phi_i)_h = \frac{1}{3} \sum_{h \in T_h: p_j \in e \cap \hat{\Omega}_j} |e| u(p_j) = |\hat{\Omega}_i| u(p_j), \quad i \in J, \quad u \in H^1_0(\Omega),
\]

\[
(f^k, \phi_i)_h = |\hat{\Omega}_i| f(p_j, t_k), \quad i \in J, \quad t_k \in [0, T].
\]

Remark 5.1.

a) from assumption (4.1-b) we find

\[
(\nabla \nabla = (\nabla \nabla .
\]

b) By \(c\) we denote a generic constant independent of \( h, \tau, k, \ldots \) which attains in general different values in different places.

6. The Maximum Principle [14]

The DMP plays an important role in computational partial differential equations by guaranteeing that approximation of physically nonnegative quantities.

Lemma 6.1. If \( \tau > 0 \) time step and \( h \in (0, h_0) \) then

\[
0 \leq \tau \leq \frac{1}{c(M) \hat{\Omega}_i}, \quad i \in J, \quad c(M) > 0.
\]

Proof. From the property (5.2-a)

\[
|e \cap \hat{\Omega}_i| = \frac{1}{3} |e| = |\hat{\Omega}_i| \Rightarrow \frac{1}{3} |e|/|\hat{\Omega}_i| = |\hat{\Omega}_i|/|\hat{\Omega}_i|,
\]

from the assumption (4.1-e) we find

\[
|e| \geq \frac{h^2}{\hat{c}} \Rightarrow \frac{1}{3} |e|/|\hat{\Omega}_i| = \frac{h^2}{3\hat{c}} |\hat{\Omega}_i|, \quad e \in T_h, \quad h \in (0, h_0),
\]

then equation (6.2) becomes

\[
|\hat{\Omega}_i|/|\hat{\Omega}_i| \geq \frac{h^2}{3\hat{c}} |\hat{\Omega}_i|,
\]

since \(|\hat{\Omega}_i| \geq h^2\), we get

\[
|\hat{\Omega}_i|/|\hat{\Omega}_i| \geq \frac{h^2}{3\hat{c}} \geq \hat{c}_i h, \quad \text{where } \hat{c}_i = 1/3\hat{c},
\]

we will consider the stability condition (see [41])

\[
0 < \tau \leq \hat{c}_i c(M)^{-1} h,
\]
then from equations (6.3) and (6.4), we obtain (6.1). \qed

**Lemma 6.2** [13]. Let \( A = (a_{ij}), C = (c_{ij}), D = (d_{ij}) \) be \( N \times K \) matrices satisfying the conditions

1. \( \sum_{j \in J} a_{ij} = \sum_{j \in J} c_{ij} = \sum_{j \in J} d_{ij} > 0 \), for \( i \in J \),
2. \( c_{ij} \geq 0 \), for \( i \in J, j \in J \),
3. \( d_{ij} \geq 0 \), for \( i \in J, j \in J \),
4. \( a_{ij} \leq 0 \), for \( i \in J, j \in J, j \neq i \),

and assume that \( A\ddot{u} = C\ddot{w} + tD\ddot{g} \),

then each component \( u_i (i \in J) \) is estimated by

\[
\max_{j \in J} |u_j| \leq \max_{j \in J} |w_j| + \tau \max_{j \in J} |g_j|.
\]

**Theorem 6.1.** If \( \tau > 0 \) and \( h \in (0, h_0) \) satisfy the condition (6.1) and if assumption (2.1-c) holds then the solution of equation (5.1) is bounded and estimated by

\[
\left\| U^k \right\|_{L^\infty(\Omega)} \leq \left\| u^0 \right\|_{L^\infty(\Omega)} + T \left\| f \right\|_{L^\infty(\partial \Omega)} \leq M.
\]

**Proof.** We prove the theorem by mathematical induction. Clearly the theorem is valid for \( k = 0 \)

\[
\left\| U^0 \right\|_{L^\infty(\Omega)} = \left\| I_h u^0 \right\|_{L^\infty(\Omega)} \leq \left\| u^0 \right\|_{L^\infty(\Omega)} \leq M,
\]

we assume that the theorem is valid for \( m \) (\( 0 \leq m \leq k \)),

\[
\left\| U^m \right\|_{L^\infty(\Omega)} \leq \left\| u^0 \right\|_{L^\infty(\Omega)} + m \tau \left\| f \right\|_{L^\infty(\partial \Omega)} \leq M.
\]

The matrix form of (5.1) given by

let \( L_h U(x,t) = \sum_{j \in J} U(p_j) \phi_j \) and choose \( \nu_h = \phi(x), \ i \in J \), one can get

\[
U^{k+1} = U^k - \tau U^k \sum_{j \in J} \phi_j \phi_i + \tau R^{k+1} \sum_{j \in J} (\lambda \nabla \phi_j, \nabla \phi_i) + \tau \sum_{j \in J} R(\ddot{B}^k, \phi_j, \phi_i) + \tau \sum_{j \in J} S(\ddot{B}^k, \phi_j, \phi_i) \phi_i \phi_j,
\]

let \( W = (w_{ij}), S = (s_{ij}) \) and \( B = (b_{ij}) \), are the matrices of mass, Stiffness and convection, respectively which have the components:

\[
w_{ij} = (\phi_j, \phi_i) = \frac{1}{3} \sum_{i \in T_h} e = \left| e \right|, \ s_{ij} = (\lambda \nabla \phi_j, \nabla \phi_i), \ b_{ij} = -R(\ddot{B}^k, \phi_j, \phi_i).
\]
Then the matrix form of equation (5.1)
\[
[W + \tau \ S] U^{k+1}(p_i) = [W + \tau \ B] U^k(p_i) + \tau \ Wf(p_i, t_{k+1}), \quad i \in J.
\]
We have [13].

\[
w_{ij} > 0 \quad w_{ij} = 0 \ (i \neq j); \quad \sum_{j \in J} s_{ij} = 0, \quad s_{ij} \leq 0 \ (i \neq j),
\]
it is easy to see

\[
(w_{ij} + \tau \ s_{ij}) \leq 0, \quad (i \in \overset{\circ}{J}, \ j \in J, \ i \neq j) \text{ and } \sum_{j \in J} (w_{ij} + \tau \ s_{ij}) > 0, \ (i \in \overset{\circ}{J}),
\]

\[
b_{ij} = -R(\tilde{B}^k, \phi_j, \phi_i)
\]

\[
= \sum_{j \in \text{ext}(i)} \sum_{a=1}^{\ell_i} \left[ (H_{ij}^k U_i^k + H_{ji}^k U_j^k) - (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \right] \beta_{ij}^{k,a}
\]
in view of the property (5.1–b) of $H$ and $H_{ij}^k + H_{ji}^k = 1$ (see [13]),

\[
\sum_{j \in \text{ext}(i)} \sum_{a=1}^{\ell_i} \left[ (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \right] \beta_{ij}^{k,a} = \sum_{j \in \text{ext}(i)} \sum_{a=1}^{\ell_i} \left[ (H_{ij}^k U_i^k + U_i^k - H_{ji}^k U_j^k) \right] \beta_{ij}^{k,a}
\]

\[
= \sum_{j \in \text{ext}(i)} \sum_{a=1}^{\ell_i} \left[ \tilde{B}^k U_i^k \alpha_b \right] \beta_{ij}^{k,a} = 0,
\]
then

\[
b_{ij} = \sum_{j \in \text{ext}(i)} \sum_{a=1}^{\ell_i} \left[ (H_{ij}^k U_i^k + H_{ji}^k U_j^k) - (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \right] \beta_{ij}^{k,a},
\]
hence, if [11]

\[
\mathcal{H}_{ij} = \begin{cases} 0, \\ \sum_{a=1}^{\ell_i} (H_{ij}^k U_i^k + H_{ji}^k U_j^k) - (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \beta_{ij}^{k,a} \end{cases} \quad U_i^k = U_j^k, \quad U_i^k \neq U_j^k
\]
then in case $U_i^k \neq U_j^k$ we get

\[
\sum_{a=1}^{\ell_i} (H_{ij}^k U_i^k + H_{ji}^k U_j^k) - (H_{ij}^k U_i^k + H_{ji}^k U_j^k) \beta_{ij}^{k,a} = \mathcal{H}_{ij}(U_j^k - U_i^k),
\]
can be written as

\[
b_{ij} = \sum_{j \in \text{ext}(i)} \mathcal{H}_{ij}(U_j^k - U_i^k),
\]
by the property (5.1–d) of $H$, $\mathcal{H}_{ij} \geq 0$, $i \in \overset{\circ}{J}$, $j \in s(i)$ then

\[
b_{ij} \geq 0.
\]
From the proof of [5, theorem 3.5.1], gives

\[
\sum_{j \in J} b_{ij} = 0, \quad i \in \overset{\circ}{J},
\]
now, we want to prove $(w_{ij} + \tau b_{ij}) \geq 0$, by the property (5.1–a) of $H$.
\[
\left| (H^k_y U^k_j + H^k_j U^k_i) - (H^k_y U^k_i + H^k_j U^k_j) \right| \leq c(M) \left( |U^k_i - U^k_j| + |U^k_j - U^k_j| \right) \\
= c(M) |U^k_i - U^k_j|,
\]

then
\[
\mathcal{H}_{ij} \leq \sum_{a=1}^{\ell_y} \frac{c(M) |U^k_i - U^k_j|}{U^k_j - U^k_i} \beta^{k,a}_y \leq c(M) \sum_{a=1}^{\ell_y} \beta^{k,a}_y,
\]
so,
\[
0 \leq \mathcal{H}_{ij} \leq c(M) \sum_{a=1}^{\ell_y} \beta^{k,a}_y = c(M) \tilde{\Gamma}_y |
\]
by using (4.1) we have,
\[
0 \leq \sum_{j \in \mathcal{I}(i)} \mathcal{H}_{ij} \leq c(M) |\tilde{\Omega}_i|,
\]
hence, the condition (6.1), gives
\[
0 \leq \sum_{j \in \mathcal{I}(i)} \mathcal{H}_{ij} \leq c(M) |\tilde{\Omega}_i| - \tau \sum_{j \in \mathcal{I}(i)} \mathcal{H}_{ij} \geq 0, \quad i \in \mathcal{J},
\]
it is easy to see that
\[
(w_{ij} + \tau b^{k}_y) = (|\tilde{\Omega}_i| - \tau \sum_{j \in \mathcal{I}(i)} \mathcal{H}_{ij} U^k_j + \tau \sum_{j \in \mathcal{I}(i)} \mathcal{H}_{ij} U^k_j) \geq 0, \quad (i \in \mathcal{J}, j \in \mathcal{J}, i \neq j),
\]
then by lemma 6.2 we get,
\[
\max_{j \in \mathcal{J}} |U^k_j| \leq \max_{j \in \mathcal{J}} |U^k_j| + \tau \max_{j \in \mathcal{J}} |f^{k+1}_j|.
\]
Since \( \|U^0\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\Omega)} \leq \gamma \), and \( \|f\|_{L^\infty(\Omega)} \leq \gamma \), by induction assumption we get
\[
\|U^{k+1}\|_{L^\infty(\Omega)} \leq \|U^0\|_{L^\infty(\Omega)} + (k + 1) \gamma.
\]

**Lemma 6.3.** If \( c^* \) is a positive constant and \( c(M) > 0 \) such that
\[
\left| R(\vec{B}^k, U^k, v) \right| \leq c^* \|v\|_{H^1_0(\Omega)}^2, \quad u \in V^h \cap L^\infty(\Omega), \quad v \in V^h, \quad h \in (0, h_0)
\]

**Proof.** By using Theorem 6.1, the property (5.1-c) of \( H \), the relations \( (\Gamma^a_{ij} = \Gamma^a_{ji} = \beta^{k,a}_{ij} = \beta^{k,a}_{ji} \) and \( \eta^{a}_{ij} = -\eta^{a}_{ji} \), (5.3) and use the property (5.1-b) of \( H \), can be get
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$$R(\bar{B}^k, U^k, v) = \left| \frac{1}{2} \sum_{i,j} \sum_{\alpha=1}^{l_H} (H^k_{ij} U_i^k + H^k_{ij} U_j^k) \beta^{k,\alpha}_{ij} \right| (v(p) - v(p_j))$$

$$= \left| \frac{1}{2} \sum_{i,j} \sum_{\alpha=1}^{l_H} (H^k_{ij} U_i^k + H^k_{ij} U_j^k) \beta^{k,\alpha}_{ij} \right| \left| (v(p) - v(p_j)) \right|$$

then by using the property 5.1-a of $H$ we find

$$\left| R(\bar{B}^k, U^k, v) \right| \leq c(M) \sum_{i,j} \sum_{\alpha=1}^{l_H} \left| U^k \right| \beta^{k,\alpha}_{ij} \left| v(p) - v(p_j) \right|,$$

since $\beta^{k,\alpha}_{ij} \leq \frac{h}{3}$ and $\left| v(p) - v(p_j) \right| \leq h$$\left| \nabla v \right|_{\Omega}'$ then

$$\left| R(\bar{B}^k, U^k, v) \right| \leq \frac{c(M)}{3} \max_{\Omega} \left| U^k \right| \sum_{i,j} \sum_{\alpha=1}^{l_H} h^2 \left| \nabla v \right|_{\Omega}'$$

from Assumption (4.1-e) and the fact that each $e \in T_h$ appears in the above sum as some $e^\alpha_{ij}$ at most six times, theorem 6.1 and from Cauchy-Schwartz inequality we conclude that

$$\left| R(\bar{B}^k, U^k, v) \right| \leq 2\hat{c}c(M) \left| U^k \right|_{L^2(\Omega)} \left| \nabla v \right|_{\Omega} dx = 2\hat{c}c(M)(\text{meas}(\Omega)^{1/2}) \left| U^k \right|_{L^2(\Omega)} \left| \nabla v \right|_{L^2(\Omega)},$$

with $c^* = 2\hat{c}c(M)(\text{meas}(\Omega)^{1/2}$.

### 7. The Stability of Approximate Solution

**Theorem 7.1.** There exists a constant $\delta > 0$ independent of $h, \tau$ and $m$ such that

$$\left\| U^m \right\|_{L^2(\Omega)} \leq \delta$$

**Proof.** In view of theorem 6.1, Assumption 2.1-c and the condition (6.4). If we set $v_h = U^{k+1}$ in (5.1) then

$$(U^{k+1} - U^k, U^{k+1})_h + \tau (\lambda \nabla U^{k+1}, \nabla U^{k+1})_h + \tau R(\bar{B}^k, U^k, U^{k+1}) = \tau (f^{k+1}, U^{k+1})_h, \tau > 0$$

we will assume the relation

$$(y - z, y)_h = \frac{1}{2} \left( \left\| y \right\|_{L^2(\Omega)}^2 - \left\| z \right\|_{L^2(\Omega)}^2 + \left\| y - z \right\|_{L^2(\Omega)}^2 \right).$$
valid for $y, z \in C(\Omega)$, we get
\[
(U^{k+1} - U^k, U^{k+1})_h = \frac{1}{2} \left( \|U^{k+1}\|^2_{L^2(\Omega)} - \|U^k\|^2_{L^2(\Omega)} + \|U^{k+1} - U^k\|^2_{L^2(\Omega)} \right),
\]
then (7.1) becomes
\[
\|U^{k+1}\|^2_{L^2(\Omega)} - \|U^k\|^2_{L^2(\Omega)} + \|U^{k+1} - U^k\|^2_{L^2(\Omega)} = 2\tau (f^{k+1}, U^{k+1})_h - 2\tau \mathcal{R}(\mathcal{B}^k, U^k, U^{k+1})_h, \quad t_k \in [0, T)
\]
(7.2)
in virtue of Lemma 6.3 and Young's inequality we have
\[
2|\mathcal{R}(\mathcal{B}^k, U^k, U^{k+1})| \leq \left( \frac{\alpha^2}{\epsilon} + \epsilon \|U^{k+1}\|^2_{H^1(\Omega)} \right),
\]
by the property (5.2-d) and Cauchy-Schwartz inequality we find
\[
|\mathcal{R}(f^{k+1}, U^{k+1})| = |(L_h f^{k+1}, L_h U^{k+1})_h| \leq \|L_h f^{k+1}\|^2_{L^2(\Omega)} \|L_h U^{k+1}\|^2_{L^2(\Omega)},
\]
now, by using the definition of the mass lamping operator $L_h$ in [13] we get
\[
\|L_h f^{k+1}\|^2_{L^2(\Omega)} = \left( \sum_{\alpha \in \Omega} \int \left( (\mathcal{B}^k(p) \mu_i(p) \right)^2 \right)^{\frac{1}{2}} \leq c^2 \|f^{k+1}\|_{C([0, T], W^{1, q}(\Omega))},
\]
(7.5)
further, we use [13, lemma 2.2] and from the definition 1.2 in [2] we get
\[
\|L_h U^{k+1}\|^2_{L^2(\Omega)} \leq \tilde{c}^2 \|U^{k+1}\|^2_{H^1(\Omega)},
\]
(7.6)
from (7.5), (7.6) and Young's inequality, we have
\[
2|\mathcal{R}(f^{k+1}, U^{k+1})| \leq \tilde{c}^2 \|f^{k+1}\|^2_{C([0, T], W^{1, q}(\Omega))} \epsilon + \epsilon \|U^{k+1}\|^2_{H^1(\Omega)},
\]
(7.7)
now, by substituting (7.3), (7.7) in (7.2), choosing $\epsilon = \frac{\lambda}{2}$ and using Remark (5.1-b), we get
\[
\|f^{k+1}\|^2_{L^2(\Omega)} - \|f^k\|^2_{L^2(\Omega)} + \|U^{k+1} - U^k\|^2_{L^2(\Omega)} + \tau \epsilon \|U^{k+1}\|^2_{H^1(\Omega)} \leq c \tau, \quad t_k \in [0, T),
\]
where $c = 2 \left( \tilde{c}^2 \|f^{k+1}\|^2_{C([0, T], W^{1, q}(\Omega))} + \epsilon \right)^{\frac{1}{2}} \lambda$. Summation over $k = 0, ..., m - 1$, $t_m \in (0, T]$ then
\[
\|U^m\|^2_{L^2(\Omega)} + \sum_{k=0}^{m-1} \|f^{k+1} - f^k\|^2_{L^2(\Omega)} + \tau \epsilon \sum_{k=1}^{m-1} \|U^{k+1}\|^2_{H^1(\Omega)} \leq cT + \|U^0\|^2_{L^2(\Omega)},
\]
(7.8)
since $U^0 = I_h u^0$ and by using the definition 1.2 in [2] then
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\[ \|U^0\|_{L^2(\Omega)}^2 \leq \tilde{c} z_2 c_1^{-2} \|I_k U^0\|_{H^1_0(\Omega)}^2, \]

then (7.8) becomes

\[ \|U^n\|_{L^2(\Omega)}^2 + \sum_{k=0}^{m_n-1} \|U^{k+1} - U^k\|_{L^2(\Omega)}^2 + \tau \sum_{k=0}^{m_n-1} \|U^{k+1}\|_{H^1_0(\Omega)}^2 \leq c T \tilde{c} z_2 c_1^{-2} \|I_k U^0\|_{H^1_0(\Omega)}^2 \delta, \quad t_m \in (0, T] \]

the last inequality is a consequence of the bounded

\[ \|I_k U^0\|_{H^1_0(\Omega)}^2 \leq \delta, \quad h \in (0, h_0) \]

and the second and third terms are nonnegative, we obtain \( \|U^n\|_{L^2(\Omega)} \leq \delta \). \( \square \)

8. The Local Truncation Error

We suppose that the exact solution \( u : (0, T) \to V \) of (2.4) satisfies the condition A(3.3).

Hence \( u \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; V^*) \cap L^\infty(Q_T) \) and set \( M = \|u^k\|_{L^\infty(Q_T)} < \infty \).

Lemma 8.1. Under conditions A(3.3), for \( t_n \in [0, T) \) we have

(a) \( \|u^{k+1} - u^k, v\| - \tau (u'(t_{k+1}), v) \| \leq c \tau \|v\|_{H^1_0(\Omega)}, \quad v \in V, \)

(b) \( \|\nabla (\tilde{b}(u^{k+1})u^{k+1}), v\| - (\nabla (\tilde{b}(u^k)u^k), v) \| \leq c \tau \|v\|_{H^1_0(\Omega)}, \quad v \in V. \)

Proof. (a) We based on the following result [10]:

If \( \omega : (0, T) \to V^* \) is such that \( \omega, \omega' \in L^1(0, T; V^*) \) and \( v \in V \) then \( \langle \omega', v \rangle \in L^1(0, T) \) and

\[ \int_{t_1}^{t_2} \langle \omega'(t), v \rangle dt = \langle \omega(t_2) - \omega(t_1), v \rangle, \quad t_1, t_2 \in [0, T]. \]

Here \( \langle \varphi, v \rangle \) denotes the value of a functional \( \varphi \in V^* \) at a point \( v \in V \). A similar result holds, if \( \omega, \omega' \in L^1(0, T; L^2(\Omega)) \) and the duality \( \langle \varphi, v \rangle \) is replaced by the \( L^2 \)-scalar product \( \langle \varphi, v \rangle \). Let \( v \in V \), then

\( (u^{k+1} - u^k, v) - \tau (u'(t_{k+1}), v) = (u(t_{k+1}) - u(t_k), v) - \int_{t_k}^{t_{k+1}} (u'(t), v) dt. \frac{t_{k+1} - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} (u'(t), v) dt \)

(8.1)

Since \( u'' \in L^\infty(0, T; V^*) \)

\( \langle u'(t) - u'(t_{k+1}), v \rangle = \int_{t_{k+1}}^t \langle u''(\omega), v \rangle d\omega, \)

(8.2)
by substituting (8.2) in (8.1) and using Cauchy-Schwartz inequality, we get
\[
\left| (u^{k+1} - u^k, v) - \tau u'(t_{k+1}), v \right| = \tau \int_{t_k}^{t_{k+1}} \left| \int_{t_k}^{t} \langle u''(\omega), v \rangle \, d\omega \right| \, dt
\]
\[
= \tau \int_{t_k}^{t_{k+1}} \frac{u'(t) - u'(t_{k+1})}{\tau}, v \right| \, dt
\]
\[
= \tau^2 \left( \frac{u'(t_{k+1}) - u'(t_k)}{\tau}, v \right) = \tau^2 \| u''(t), v \| \leq c \tau^2 \| u''(t), v \|_{H^2_0(\Omega)};
\]

(b) By using Green's theorem then
\[
\left| (\nabla (\overline{b}(u^{k+1}), u^{k+1}), v) - (\nabla (\overline{b}(u^k), u^k), v) \right| = \left| (\overline{b}(u^{k+1}), u^{k+1}, \nabla v) - (\overline{b}(u^k), u^k, \nabla v) \right|
\]
adding and subtracting \((\overline{b}(u^{k+1}), u^k, \nabla v)\), by using Cauchy-Schwartz inequality, Theorem 8.1 and the conditions A(3.1)–A(3.3), we get
\[
\left| (\nabla (\overline{b}(u^{k+1}), u^{k+1}), v) - (\nabla (\overline{b}(u^k), u^k), v) \right| \leq \left| (\overline{b}(u^{k+1}), (u^{k+1} - u^k), \nabla v) + ((\overline{b}(u^{k+1}) - \overline{b}(u^k))u^k, \nabla v) \right|
\]
\[
\leq \frac{c}{\tau} \| u^{k+1} - u^k \|_{L^2(\Omega)} \| v \|_{H^0_0(\Omega)},
\]
since \(u' \in L^\infty (0, T; L^2 (\Omega))\)
\[
\| u^{k+1} - u^k \|_{L^2(\Omega)} \leq \tau \| u' \|_{L^\infty (0, T; L^2 (\Omega))} \leq c \tau ,
\]
then (8.3) becomes
\[
\left| (\nabla (\overline{b}(u^{k+1}), u^{k+1}), v) - (\nabla (\overline{b}(u^k), u^k), v) \right| \leq c \tau \| v \|_{H^0_0(\Omega)} , \quad v \in V .
\]

Theorem 8.2. If \(u^k\) and \(U^k\) are solutions of (2.4) and (5.1), respectively at time \(t = t_k\) and \(u^k\) satisfies the conditions A(3.3) then
\[
\text{L.T.E} \leq 0(\tau^2), \quad \tau > 0 .
\]

Proof. The exact solution \(u\) satisfies (2.4) at \(t = t_{k+1}\) then
\[
(u'(t_{k+1}), v) + (\lambda \nabla u^{k+1}, \nabla v) + (\nabla (\overline{b}(u^{k+1}), u^{k+1}, \nabla v) = (f^{k+1}, v) , \quad \forall v \in V ,
\]
adding and subtracting \((\frac{u^{k+1} - u^k}{\tau}, v)\) and \((\nabla (\overline{b}(u^k), u^k), v)\) with setting \(v = v_h \in V_h\), gives
\[
(u^{k+1} - u^k, v_h) + \tau (\lambda \nabla u^{k+1}, \nabla v_h) + \tau (\nabla (\overline{b}(u^k), u^k), v_h) = \tau (f^{k+1}, v_h) +
\[
[(u^{k+1} - u^k, v_h) - (u'(t_{k+1}), v_h)] + \tau (\nabla (\overline{b}(u^k), u^k), v_h) - (\nabla (\overline{b}(u^{k+1}), u^{k+1}), v_h) | v_h \in V_h ,
\]
by subtracting equation (8.4) from the equation (5.1) and using Remark (5.1-a) such that
(U^{k+1} - U^k, v_h) - (u^{k+1} - u^k, v_h) + \tau[(A\nabla U^{k+1}, \nabla v_h) - (A\nabla u^{k+1}, \nabla v_h)]

+ \tau[R(\tilde{b}, U^k, v_h) - (\nabla.(\tilde{b}(u^k)) u^k, v_h)]

- [(u^{k+1} - u^k, v_h) - \tau(u'(t_{k+1}), v_h)] - \tau[(\nabla.(\tilde{b}(u^k)) u^k, v_h) - (\nabla.(\tilde{b}(u^{k+1})) u^{k+1}, v_h)], \quad \forall v_h \in V_h

adding and subtracting (U^{k+1} - U^k, v_h) and \tau(\nabla.(\tilde{b}(U^k)).U^k), v_h), gives

\begin{align}
(U^{k+1} - u^{k+1}, v_h) - (u^k - u^k, v_h) + \tau(\nabla(U^{k+1} - u^{k+1}), \nabla v_h) &

+ \tau[(\nabla.(\tilde{b}(U^k)).U^k), v_h) - (\nabla.(\tilde{b}(u^k)) u^k, v_h)] = I_1 + I_2, \quad \forall v_h \in V_h
\end{align}

where

\begin{align}
I_1 &= - [(u^{k+1} - u^k, v_h) - \tau(u'(t_{k+1}), v_h)]

- \tau[(\nabla.(\tilde{b}(u^k)) u^k, v_h) - (\nabla.(\tilde{b}(u^{k+1})) u^{k+1}, v_h)],

I_2 &= \tau[(f^{k+1}, v_h) - (f^k, v_h)] + [(U^{k+1}, v_h) - (U^k, v_h)]

- [(U^k, v_h) - (U^k, v_h)] + \tau[(\nabla.(\tilde{b}(U^k)).U^k), v_h) - R(\tilde{b}, U^k, v_h)].
\end{align}

To estimate \( I_1 \), we use Lemma 8.1,

\begin{align}
|I_1| &\leq \|u^{k+1} - u^k, v_h) - \tau(u'(t_{k+1}), v_h)] + \tau\|[(\nabla.(\tilde{b}(u^k)) u^k, v_h) - (\nabla.(\tilde{b}(u^{k+1})) u^{k+1}, v_h)]

&\leq c\tau^2\|v_h\|_{H^1_0(\Omega)} \leq 0(\tau^2), \quad v_h \in V_h, \quad \tau > 0.
\end{align}

We can write \( I_2 = \sum_{i=1}^{\delta} I_{2i} \). The estimate \( I_{21} \) is a consequence of [ 12, Theorem 4.1.5 and the proof of Theorem 4.1.6 ]

\begin{align}
|I_{21}| = \tau[(f^{k+1}, v_h) - (f^k, v_h)] \leq \tau h\|v_h\|_{H^1_0(\Omega)}, \quad \forall v_h \in V_h.
\end{align}

The estimate of \( I_{22} \) and \( I_{23} \) are a consequence of [ 12, Theorem 3.1.6 ] and using Lemma 2.1, we have

\begin{align}
|I_{22}| = \|U^{k+1}, v_h] - (U^k, v_h)) \leq c h^2\|v_h\|_{H^1_0(\Omega)},

\end{align}

so we find

\begin{align}
|I_{22}| = \|U^k, v_h] - (U^k, v_h)) \leq c h^2\|v_h\|_{H^1_0(\Omega)}.
\end{align}

To estimate \( I_{24} \), by adding and subtracting \( \tau(\nabla.(\tilde{b}(U^k)).U^k), L_\alpha v_h) \) then

\begin{align}
|I_{24}| = \tau\|[(\nabla.(\tilde{b}(U^k)) u^k, v_h) - R(\tilde{b}, U^k, v_h)]

\end{align}
\[ \tau \left( \nabla \cdot (\overline{b}(U^k).U^k),v_h \right) - \left( \nabla \cdot (\overline{b}(U^k).U^k),L_h v_h \right) \leq \frac{1}{2} \sum_{e \in \mathcal{T}_h} h^3 c(M) \| \nabla U^k \|_{\Omega} \| \nabla v_h \|_{\Omega} \].

(8.10)

If \( i \in J \) and \( j \in s(i) \), then the segment \( p_i p_j \) a side of triangles \( e_{ij}^a \in \mathcal{T}_h \) such that \( \Gamma_{ij}^a \subseteq e_{ij}^a \) then.

\[ |p_i - p_j| \leq h , \quad |x - p_i| \leq \frac{h}{3} \quad \text{for } x \in e_{ij}^a \), \quad \beta_{ij}^{k,a} \leq \frac{h}{3} , \quad \| u^k(p_i) - u^k(p_j) \| \leq h \| \nabla u^k \|_{\Omega} , \]

\[ |u^k(x) - u^k(p_i)| \leq h \| \nabla u^k \|_{\Omega} \quad \text{for } x \in e_{ij}^a \), \quad \| v_h(p_i) - v_h(p_j) \| \leq h \| \nabla v_h \|_{\Omega} , \]

(8.11)

by the properties (5.1-a,b) of \( \mathcal{H} \) and (8.11), we obtain

\[ \left\| \overline{b}(U^k) U^k n_{ij}^a ds - (H_{ij} U^k + H_{ij} U^k) \beta_{ij}^{k,a} \right\| \leq 2c(M) \max_{x \in e_{ij}^a} |U^k(x) - U^k| \| \beta_{ij}^{k,a} \| + c(M) |U^k - U^k| \| \beta_{ij}^{k,a} \| \leq h^2 c(M) \| \nabla U^k \|_{\Omega} , \]

then we have

\[ \tau \left( \nabla \cdot (\overline{b}(U^k).U^k),v_h \right) - R(\overline{B}^k, U^k, v_h) \leq \frac{1}{2} \sum_{e \in \mathcal{T}_h} h^3 c(M) \| \nabla U^k \|_{\Omega} \| \nabla v_h \|_{\Omega} \],

(8.12)

taking into account that each triangle \( e \in \mathcal{T}_h \) appears in the sum in (8.12) as some \( e_{ij}^a \) at most six times, by using Assumption (4.1-e), Cauchy-Schwartz inequality and Lemma 2.1 we conclude that

\[ \tau \left( \nabla \cdot (\overline{b}(U^k).U^k),v_h \right) - R(\overline{B}^k, U^k, v_h) \leq \hat{c}_\tau(M) \tau h \sum_{e \in \mathcal{T}_h} \| \nabla U^k \|_{\Omega} \| \nabla v_h \| \leq \tau h \| v_h \|_{H^1(\Omega)} , \]

(8.13)

now from (8.10) and (8.13) we get

\[ |I_{24}| \leq \tau h \| v_h \|_{H^1(\Omega)} , \]
by using Assumption (4.1-f) and from (8.7)-(8.9), (8.14) we get
\[ | f_2 | \leq 0(\tau^2), \tag{8.15} \]
and from (8.6) and (8.15) the proof is complete.

9. The Error Estimate

We define the projection \( p_h : V \to V_h \) as follows: If \( \varphi \in V \) such that \( p_h \varphi \in V_h \),
\[ \left| p_h \varphi \right|_{H^1_0(\Omega)} \leq \left| \varphi \right|_{H^1_0(\Omega)}, \quad \varphi \in V. \tag{9.1} \]

**Lemma 9.1.** For any \( M > 0 \) there exists a constant \( \xi > 0 \) such that
\[ \left| (\nabla \cdot (\tilde{b}(z_1), z_1), v) - (\nabla \cdot (\tilde{b}(z_2), z_2), v) \right| \leq \xi \left| z_1 - z_2 \right|_{L^2(\Omega)}, \]
\( z_1, z_2 \in H^1_0(\Omega) \cap L^\infty(\Omega), \left| z_1 \right|_{L^\infty(\Omega)}, \left| z_2 \right|_{L^\infty(\Omega)} \leq M. \)

**Proof.** Is similar to proof Lemma 8.1-b.

**Theorem 9.1.** If \( u^k \) and \( U^k \) are solutions of (2.4) and (5.1) respectively and satisfy the conditions \( A(3.3) \) and \( \tau \) be the time step satisfy \( 0 \leq \tau \leq \left| \bar{\Omega} \right|/c(M)\left| \bar{\Omega} \right|, \quad i \in J \) then
\[
\max_{t_i \in [0,T]} \left| e^i \right|_{L^2(\Omega)} \leq 0(\tau), \tag{9.2}
\]
where a constant \( c > 0 \) is independent of \( h, \tau, \lambda \).

**Proof.** Let \( h \in (0, h_o) \) and \( \tau > 0 \) satisfy the condition (7.4) and Assumption (4.1-f), From (8.5), let the error of the method at time \( t = t_k \) is defined by
\[ e^k = U^k - u^k, \tag{9.3} \]
let \( v_h = p_h e^{k+1} \), adding and subtracting \((e^{k+1}, e^{k+1}), (e^k, e^{k+1})\) and \( \tau \lambda (\nabla e^{k+1}, \nabla e^{k+1}), \) gives
\[
(e^{k+1}, e^{k+1}) - (e^k, e^{k+1}) + \tau \lambda (\nabla e^{k+1}, \nabla e^{k+1}) =
\]
\[
- \tau \left[ (\nabla \cdot (\tilde{b}(U^k), U^k), p_h e^{k+1}) - (\nabla \cdot (\tilde{b}(u^k), u^k), p_h e^{k+1}) \right] + I_1 + I_2 + (e^{k+1}, e^{k+1})
\]
\[
- (e^{k+1}, p_h e^{k+1}) - (e^k, p_h e^{k+1}) + (e^k, p_h e^{k+1}) + \tau \lambda (\nabla e^{k+1}, \nabla e^{k+1}) - \tau \lambda (\nabla e^{k+1}, p_h e^{k+1})
\]
denoting by \( I : V \to V \) the identity operator \( (I \varphi = \varphi \text{ for } \varphi \in V) \) then
\[
\left(e^{k+1} - e^k, e^{k+1}\right) + \tau\lambda\left(\nabla e^{k+1}, \nabla e^{k+1}\right) = -\tau\{(\nabla(\tilde{b}(U^k)U^k)), p_\delta e^{k+1}\} - \left(\nabla(\tilde{b}(u^k)u^k)), p_\delta e^{k+1}\right) + I_1 + I_2 + (e^{k+1}, (I - p_\delta)e^{k+1}) - (e^k, (I - p_\delta)e^{k+1}) - \tau\lambda(\nabla e^{k+1}, \nabla(I - p_\delta)e^{k+1}),
\]

(9.4)

from (9.3) it follows that

\[
(I - p_\delta)e^{k+1} = U^{k+1} - u^{k+1} - p_\delta U^{k+1} + p_\delta u^{k+1} = p_\delta u^{k+1} - u^{k+1},
\]

(9.5)

hence, by using Lemma 2.2, Lemma 2.3 and (9.5) then

\[
\|I - p_\delta\|_{L^2(\Omega)} \leq ch^2\|\tilde{\phi}^k\|_{H^1(\Omega)}
\]

(9.6)

\[
\|I - p_\delta\|_{L^2(\Omega)} \leq ch\|\tilde{\phi}^k\|_{L^2(\Omega)}
\]

(9.7)

by using the relation \(A.B = \frac{1}{4}[(A + B)^2 - (A - B)^2]\) and Cauchy-Schwartz inequality, we find

\[
(e^{k+1} - e^k, e^{k+1}) = \frac{1}{2}\left(\|\tilde{\phi}^k\|^2_{L^2(\Omega)} + \|e^{k+1} - e^k\|^2_{L^2(\Omega)} - \|e^k\|^2_{L^2(\Omega)}\right).
\]

(9.8)

And

\[
\left(\nabla e^{k+1}, \nabla e^{k+1}\right) = \left|e^{k+1}\right|_{H^1(\Omega)}^2,
\]

(9.9)

it follows from Lemma 9.1, (9.1) and (9.3) that

\[
\left(\nabla(\tilde{b}(U^k)U^k)), p_\delta e^{k+1}\right) - \left(\nabla(\tilde{b}(u^k)u^k)), p_\delta e^{k+1}\right) \leq c\|e^k\|_{L^2(\Omega)}\|e^{k+1}\|_{H^1(\Omega)},
\]

(9.10)

and since

\[
|I_1| \leq c\tau^2\|e^{k+1}\|_{H^1(\Omega)},
\]

(9.11)

\[
|I_2| \leq c\tau h\|\tilde{\phi}^k\|_{H^1(\Omega)} + c\tau h\|\tilde{\phi}^k\|_{H^1(\Omega)}
\]

(9.12)

furthermore, by using Cauchy-Schwartz inequality and (9.6), we have

\[
\left|e^{k+1}, (I - p_\delta)e^{k+1}\right| - \left(e^k, (I - p_\delta)e^{k+1}\right) \leq ch^2\|e^{k+1} - e^k\|^2_{L^2(\Omega)} + c\|e^k\|^2_{L^2(\Omega)}\|e^{k+1}\|_{H^2(\Omega)},
\]

(9.13)

so, by using Cauchy-Schwartz inequality, from the definition 1.2 in [2] and (9.6), we have

\[
\left(\nabla e^{k+1}, \nabla(I - p_\delta)e^{k+1}\right) \leq ch\|\tilde{\phi}^k\|_{H^1(\Omega)}\|e^{k+1}\|_{H^2(\Omega)},
\]

(9.14)

now, by substituting (9.8)–(9.14) in (9.4). Taking into account conditions A(3.3) we obtain the estimate

\[
\|e^{k+1}\|^2_{L^2(\Omega)} + \|e^{k+1} - e^k\|^2_{L^2(\Omega)} - \|e^k\|^2_{L^2(\Omega)} + 2\tau\lambda\|e^{k+1}\|^2_{H^1(\Omega)} \leq c\tau\|e^k\|^2_{L^2(\Omega)}\|e^{k+1}\|_{H^1(\Omega)}.
\]
\[ + c \tau^2 \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2 + \tau h \| e^{k+1}_\Omega \|_{l^2(\Omega)} + ch^2 \| e^{k+1}_\Omega \|_{l^2(\Omega)} + 2ch^2 \| e^{k+1}_\Omega - e^k \|_{l^2(\Omega)} + \tau \lambda h \| e^{k+1}_\Omega \|_{l^2(\Omega)}, \]

We use (6.4) and assumption (4.1-f) then
\[ \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2 + \| e^{k+1}_\Omega - e^k \|_{l^2(\Omega)}^2 - \| e^k \|_{l^2(\Omega)}^2 + 2\tau \lambda \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2 \leq c \tau \| e^k \|_{l^2(\Omega)} + c \| e^{k+1}_\Omega \|_{l^2(\Omega)} + \| e^{k+1}_\Omega - e^k \|_{l^2(\Omega)} + \tau \lambda h \| e^{k+1}_\Omega \|_{l^2(\Omega)}, \]

by using Young's inequalities we find that
\[ \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2 \leq A \| e^k \|_{l^2(\Omega)}^2 + \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2 + \| e^{k+1}_\Omega - e^k \|_{l^2(\Omega)}^2 + \tau \lambda h \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2, \]

and by using Assumption (4.1-f), since the third term is nonnegative we have
\[ \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2 \leq A \| e^k \|_{l^2(\Omega)}^2 + \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2 + \| e^{k+1}_\Omega - e^k \|_{l^2(\Omega)}^2 + \tau \lambda h \| e^{k+1}_\Omega \|_{l^2(\Omega)}^2, \]

where
\[ A = 1 + \frac{c \tau}{\lambda}, \quad B = c \left[ \frac{h^2}{\lambda} (1 + \lambda^2) + h^3 \right], \]

by induction over \( k = 0, 1, 2, \ldots \), from (9.15) we easily deduce that
\[ \| e^k \|_{l^2(\Omega)}^2 \leq A^k \| e^0 \|_{l^2(\Omega)}^2 + \tau (A^{k-1} + A^{k-2} + \ldots + 1)B, \]

multiplying and dividing the second term by \( (A - 1) \)
\[ \| e^k \|_{l^2(\Omega)}^2 \leq A^k \| e^0 \|_{l^2(\Omega)}^2 + \tau B \frac{A^k - 1}{A - 1}, \quad t_k \in [0, T), \]

since \( A \leq \exp(\frac{c \tau}{\lambda}) \), then
\[ \| e^k \|_{l^2(\Omega)}^2 \leq \exp\left(\frac{cl_k}{\lambda}\right) \| e^0 \|_{l^2(\Omega)}^2 + c \left[ \frac{h^2}{\lambda} (1 + \lambda^2) + h^3 \right] \left(\exp\left(\frac{cl_k}{\lambda}\right) - 1\right) \lambda, \]

taking into account that \( u^e \in H^1(\Omega) \subset W^{1,p}_{0}(\Omega) \), by virtue of \([ 8, \text{Theorem 2.27} ]\) and the conditions A(3.3) we have
\[ \| e^k \|_{l^2(\Omega)}^2 \leq ch^2 \| u^0 \|_{l^2(\Omega)}, \]

also we obtain the estimate
\[ \| e^k \|_{l^2(\Omega)} \leq c \exp\left(\frac{cl_k}{2\lambda}\right) h + c \left[ h(1 + \lambda^2)^{1/2} + \sqrt{\lambda h^{3/2}} \right] \left(\exp\left(\frac{cl_k}{\lambda}\right) - 1\right)^{1/2}, \quad t_k \in [0, T], \]
now \( \max_{\lambda \in [0,1)} \| e^A \|_{L^2(\Omega)} \leq \left( c \exp\left( \frac{cT}{\lambda} \right) + c(1 + \lambda) \exp\left( \frac{cT}{\lambda} \right) \right) h + c\sqrt{\lambda} h^{\frac{3}{2}} \exp\left( \frac{cT}{\lambda} \right), \)

finally, by using Assumption (4.1-f) , which immediately implies (9.2). \( \square \)

Conclusions

In this paper we have analyzed the U.F.V. method for solving nonlinear convection-diffusion equation. This method satisfies the convergence, the discrete maximum principle and the discrete conservation law. We derive the U.F.V. method for the convection term \( \nabla (\mathbf{b}(u) \cdot \mathbf{u}) \) which is discretized on the barycentric finite volume mesh dual to a triangular grid of weakly acute type under some assumptions on the regularity of the exact solution of the continuous problem. The discrete maximum principle implying the \( L^\infty \) -estimate and the stability of approximate solutions are the basic tools used in the investigation of error estimate are proved under the condition \( 0 \leq \tau \leq \frac{\| \mathbf{b} \|_{L^\infty(\Omega)}}{M} \frac{\| \partial_i \mathbf{u} \|_{L^\infty(\Omega)}}{M} \), \( i \in J \).

References


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