Stability Analysis and Static Output Feedback Design for a Model the Fishery Resource with Reserve Area

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Abstract

In this paper, the problem of global state regulation via output feedback is investigated to study a fishery resource system, in order to stabilize its states around a non-trivial equilibrium. In our case, the two zones: a free fishing zone and a reserve zone where fishing is strictly prohibited. In order to apply the tools of automatic control to our model, the fishing effort is used as a control term, the age classes as a states and the quantity of captured fish per unit of effort as a measured output. A Lyapunov is adapted to study the stability and stabilization of the studied system around the non-trivial steady states. Numerical simulation demonstrated the effectiveness and the convergence of the states to the equilibrium.

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1 Introduction

In a renewable resource (e.g. fishery and forestry) management modeling was motivated by the need to understand mechanisms governing production flows
of marine reserves. Several models were built and their analyses helped to identify management measures adapted to specific objectives (see [2],[3],[6],[7],[8]). The control theory can be used to address the problem of defining a good harvesting policy, by stabilizing the stock states around the references equilibrium, which means biologically the sustainability of the fish stock. When solving this control engineering problem, it is often necessary to know the state of a dynamical system. But in fishery systems the states variables can’t be measured and the resources can’t be counted directly except with acoustic method which is not generalized yet. Therefore, the presence of unknown states becomes a difficulty which can be solved by means of the inclusion of output feedback control.

Here we consider the dynamics of a fish population moving between two zones. The first zone is a free fishing area, and the second zone is a reserve area where no fishing is allowed. This model is given by systems of differential equations of the form:

$$\dot{X} = F(X, E) \quad (1)$$

where $E$ is the fishing effort (it can be seen as a control or an input) and $X(t)$ is the state of the system at time $t$. The state variable $X(t)$ represents the density of the population or the number of individuals. This model is inspired from the classical model:

$$\dot{N}(t) = \text{births} - \text{deaths} + \text{migration} \quad (2)$$

when $N(t)$ represents the density of the population or the number of individuals.

To make a policy decision about the exploitation of renewable resources, it is necessary to take into account the state of the resource stocks. However the states $X(t)$ are not available. We can only measure the total catch at each time $t$. The total catch per unit of effort is regarded as output of system (1).

The remainder of the paper is organized as follows: Section 2 gives a brief exposition of the model. In Section 3, a Lyapunov function is adapted to study the stabilization of the studied system around the nontrivial steady states. In Section 4, we show that the system can be regulated with a linear feedback control that is extended to an output feedback one. Then a numerical example is presented to demonstrate the effectiveness and the convergence of the states to the equilibrium. Finally a conclusion is given.

## 2 The Model and Preliminaries

Consider a fish habitat in an aquatic ecosystem, which will be left with two areas reserved for the first and the second for non-reserved. Modeling system assumes that fishing is allowed in the restricted area while the area is an area
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fully open access fishing. Let \( x_1(t) \) and \( x_2(t) \) the density of the respective biomass of the fish population within the same unreserved and reserved areas, respectively, at time \( t \). Let \( E \) be the total effort applied for harvesting the fish population in the unreserved area is the subset of fish area fully migrate into an area reserved for a rate \( \sigma_1 \) as the sub-population of fish of the reserved area migrate into the area without reservation to a rate \( \sigma_2 \).

It is assumed that growth of the two sub-population in each zone reacts according to the logistic model. The dynamics of fish populations to da full and reserved areas are then supposed to be described by the autonomous system of differential equations following (see [3]):

\[
\begin{align*}
\dot{x}_1 &= rx_1 \left(1 - \frac{x_1}{K}\right) - \sigma_1 x_1 + \sigma_2 x_2 - qE x_1, \\
\dot{x}_2 &= sx_2 \left(1 - \frac{x_2}{L}\right) + \sigma_1 x_1 - \sigma_2 x_2, \\
x_1(0) &> 0, x_2(0) > 0.
\end{align*}
\] (3)

Here \( r \) and \( s \) represent the intrinsic growth of each fish sub-population, respectively, \( K \) and \( L \) are the carrying capacities of fish species in the unreserved and reserved areas, respectively, \( q \) is the catchability coefficient of fish species in the unreserved area.

The parameters \( r, s, q, \sigma_1, \sigma_2, K \) and \( L \) are positives constants.

We note that for a migration of fish populations in the area reserved for fully and vice versa, we assume that:

\[
r - \sigma_1 - qE > 0, \quad s - \sigma_2 > 0
\] (4)

2.1 Equilibrium point

Equilibria of the system (3) is obtained by \( \dot{x}_1 = \dot{x}_2 = 0 \). It can be checked that model (3) has only two nonnegative equilibria \((0, 0)\) and \( x^* = (x_1^*, x_2^*) \). Where \( x_1^* \) and \( x_2^* \) are the positive solutions of the following equations:

\[
\begin{align*}
\sigma_2 x_2 &= \frac{r x_1^2}{K} - (r - \sigma_1 - qE) x_1, \\
\sigma_1 x_1 &= (\sigma_2 - s) x_2 + \frac{sx_2^2}{L}
\end{align*}
\] (5)

By replacing the value of \( x_2 \) in the second equation, we obtain the equation in terms of \( x_1 \) as follows:

\[
a x_1^3 + b x_1^2 + a x_1 + d = 0
\]
where

\[
\begin{align*}
    a &= \frac{sr^2}{L\sigma_2^2 K^2}, \\
    b &= -\frac{2sr(r - \sigma_1 - qE)}{L\sigma_2^2 K}, \\
    c &= \frac{s(r - \sigma_1 - qE)^2}{L\sigma_2^2} - \frac{(s - \sigma_2)r}{K\sigma_2}, \\
    d &= \frac{(s - \sigma_2)}{\sigma_2} (r - \sigma_1 - qE) - \sigma_1.
\end{align*}
\]

The above equation has a unique positive solution \( x_1 = x_1^* \) if the following inequalities hold:

\[
\frac{s(r - \sigma_1 - qE)^2}{L\sigma_2} < \frac{(s - \sigma_2)r}{K} \\
(s - \sigma_2)(r - \sigma_1 - qE) < \sigma_1\sigma_2
\]

(6)

Knowing the value of \( x_1^* \), the value of \( x_2^* \) can then be computed from (5). It may be noted here that for \( x_2^* \) to be positive, we must have

\[
\frac{rx_1^*}{K} > r - \sigma_1 - qE
\]

(7)

2.2 Boundedness

In the following lemma we show that all solutions of model (3) are positive and uniformly bounded.

**Lemma 2.1** (see [3]) All the solutions of the system (3) which start in \( \mathbb{R}^2_+ \) is in the compact:

\[
\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \leq \frac{\mu}{\eta} \right\}
\]

(8)

where \( \eta \) is a positive constant and

\[
\mu = \frac{K}{4r} (r + \eta - qE)^2 + \frac{L}{4s} (s + \eta)^2
\]

3 Stability Analysis

According to a formulation due to Capasso (see [1]) the above model can be written in the general form:

\[
\dot{x} = \text{diag}(x) \left( A_0 + A_1 x \right) + Bx
\]

(9)

where \( A_0 = \begin{pmatrix} r - \sigma_1 - qE \\ s - \sigma_2 \end{pmatrix} \), \( A_1 = \begin{pmatrix} -\frac{r}{K} & 0 \\ 0 & -\frac{s}{L} \end{pmatrix} \).
and $B = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_1 & 0 \end{pmatrix}$

The equilibrium point $x^* \in \Omega$ satisfies the above equation:

$$\text{diag}(x^*)(A_0 + A_1x^*) + Bx^* = 0$$

which implies

$$A_0 = -A_1x^* - \text{diag} \left( \frac{1}{x^*} \right) Bx^*$$

by substitution in (9), we obtain

$$\dot{x} = \text{diag}(x) \left( A + \text{diag} \left( \frac{1}{x^*} \right) B \right) (x - x^*) - \text{diag}(x - x^*) \text{diag} \left( \frac{1}{x^*} \right) Bx$$ \hspace{1cm} (10)

diag(.) is a diagonal matrix.

**Theorem 3.1** For any constant fishing effort $E$, under the assumptions (5), (6) and (7). $x^*$ is globally asymptotically stable.

**Proof 3.2** Let $V$ a candidate Lyapunov function for system (3) (see [1],[4]):

$$V(x) = \omega_1 \left( x_1 - x_1^* - x_1^* \ln \left( \frac{x_1}{x_1^*} \right) \right) + \omega_2 \left( x_2 - x_2^* - x_2^* \ln \left( \frac{x_2}{x_2^*} \right) \right)$$ \hspace{1cm} (11)

where $\omega_i \ i = 1, 2$ are strictly positive coefficients.

The time derivative of $V$ computed along solutions of the differential equations (3) is:

$$\dot{V}(x) = (x - x^*)^T W \tilde{A}(x - x^*) - \sum_{i=1}^{2} \omega_i \frac{(Bx)_i}{x_i x_i^*} (x_i - x_i^*)^2$$ \hspace{1cm} (12)

where

$$\tilde{A} = A + \text{diag} \left( \frac{1}{x^*} \right) B = \begin{pmatrix} \frac{r}{K} & \frac{\sigma_2}{x_1^*} \\ \frac{\sigma_1}{x_2^*} & -s \frac{\sigma_1 x_1}{L} \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

The (12) becomes:

$$\dot{V}(x) = (x - x^*)^T W \tilde{A} (x - x^*)$$ \hspace{1cm} (13)

where

$$\tilde{A} = \begin{pmatrix} \frac{r}{K} - \frac{\sigma_2 x_2}{x_1 x_1^*} & \frac{\sigma_2}{x_1 x_1^*} \\ \frac{\sigma_1}{x_2^*} & -s \frac{\sigma_1 x_1}{L} - \frac{\sigma_1 x_1}{x_2 x_2^*} \end{pmatrix}$$
It follows that sufficient conditions for the system (3) to be globally stable is $W(-\tilde{A}_1) + (-\tilde{A}_1)^\top W$ to be positively definite, which is satisfied if the determinants of their principal minors $\Delta_1$ and $\Delta_2$ are positives.

we have

$$W \tilde{A}_1 = \begin{pmatrix} \omega_1 \left( \frac{r}{K} - \frac{\sigma_2 x_2}{x_1 x_1^*} \right) & \omega_1 \left( \frac{\sigma_2}{x_1^*} \right) \\ \omega_2 \left( \frac{\sigma_1}{x_2^*} \right) & -\omega_2 \left( \frac{s}{L} - \frac{\sigma_1 x_1}{x_2 x_2^*} \right) \end{pmatrix}$$

then

$$W(-\tilde{A}_1) + (-\tilde{A}_1)^\top W = \begin{pmatrix} 2\omega_1 \left( \frac{r}{K} + \frac{\sigma_2 x_2}{x_1 x_1^*} \right) & -\omega_1 \left( \frac{\sigma_2}{x_1^*} \right) - \omega_2 \left( \frac{\sigma_1}{x_2^*} \right) \\ -\omega_1 \left( \frac{\sigma_2}{x_1^*} \right) - \omega_2 \left( \frac{\sigma_1}{x_2^*} \right) & 2\omega_2 \left( \frac{s}{L} + \frac{\sigma_1 x_1}{x_2 x_2^*} \right) \end{pmatrix}$$

The principal minors are:

$$\Delta_1 = 2\omega_1 \left( \frac{r}{K} + \frac{\sigma_2 x_2}{x_1 x_1^*} \right),$$

and

$$\Delta_2 = 4\omega_1 \omega_2 \left( \frac{r}{K} + \frac{\sigma_2 x_2}{x_1 x_1^*} \right) \left( \frac{s}{L} + \frac{\sigma_1 x_1}{x_2 x_2^*} \right) - \left( \omega_1 \left( \frac{\sigma_2}{x_1^*} \right) + \omega_2 \left( \frac{\sigma_1}{x_2^*} \right) \right)^2$$

$$= -\omega_1^2 \left( \frac{\sigma_2}{x_1^*} \right)^2 + 2\omega_1 \omega_2 \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} + 2 \frac{rs}{KL} - \omega_2^2 \left( \frac{\sigma_1}{x_2^*} \right)^2 \quad (14)$$

In order to guarantee the positivity of $\Delta_2$ it is enough to find $\omega_2$ for a known $\omega_1$ such that:

$$-\omega_1^2 \left( \frac{\sigma_2}{x_1^*} \right)^2 + 2\omega_1 \omega_2 \left( \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} + 2 \frac{rs}{KL} \right) - \omega_2^2 \left( \frac{\sigma_1}{x_2^*} \right)^2 > 0$$

which equivalent to

$$-\omega_1^2 \left( \frac{\sigma_2}{x_1^*} \right)^2 + 2\omega_1 \omega_2 \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} + 2 \frac{rs}{KL} - \omega_2^2 \left( \frac{\sigma_1}{x_2^*} \right)^2 > 0$$

Let us pose $\lambda = \frac{\omega_1}{\omega_2}$ and $\Delta$ the corresponding discriminant for the above equation:

$$- \left( \frac{\sigma_2}{x_1^*} \right)^2 \lambda^2 + 2 \left( \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} + 2 \frac{rs}{KL} \right) \lambda - \left( \frac{\sigma_1}{x_2^*} \right)^2 = 0 \quad (15)$$
we have

\[ \Delta = \left( \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} + 2 \frac{rs}{KL} \right)^2 - \left( \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} \right)^2 > 0 \]

So the equation (15) admits two solutions:

\[ \lambda_1 = \frac{-\left( \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} + 2 \frac{rs}{KL} \right) + \sqrt{\Delta}}{-\left( \frac{\sigma_1}{x_1^*} \right)^2} > 0 \]

\[ \lambda_2 = \frac{-\left( \frac{\sigma_1 \sigma_2}{x_1^* x_2^*} + 2 \frac{rs}{KL} \right) - \sqrt{\Delta}}{-\left( \frac{\sigma_1}{x_2^*} \right)^2} > 0 \]

for a known \( \omega_1 \) and \( \lambda \) belonging to the interval \([\lambda_1, \lambda_2]\) the inequality is satisfied. Hence \( \Delta_2 \) in the equation (14) is positive. Clearly it is possible to choose positive numbers \( \omega_1 \) and \( \omega_2 \) so that \( \Delta_1 \) and \( \Delta_2 \) are positive.

4 Output Feedback Control

The simple fishery dynamic system considered constituted of the juvenile and adults as states and the fishing effort \( u = E \) as the input. The fishing effort generated by the fleet is applied to the stock and produces a catch of \( y \) per unit effort.

The problem addressed here is to construct an output feedback control \( E(t) = \bar{E} + u(y(t)) \) in such a manner that the state \( x^* \) is a globally asymptotically stable equilibrium point for the closed-loop system. The system (3) can be written in canonical form:

\[ \dot{x} = f(x) + ug(x) \] (16)

where

\[ f(x) = \begin{pmatrix} rx_1 \left( 1 - \frac{x_1}{K} \right) - \sigma_1 x_1 + \sigma_2 x_2 \\ sx_2 \left( 1 - \frac{x_2}{L} \right) + \sigma_1 x_1 - \sigma_2 x_2 \end{pmatrix} \quad \text{and} \quad g(x) = \begin{pmatrix} -qx_1 \\ 0 \end{pmatrix} \]

**Theorem 4.1** For any constant fishing effort \( \bar{E} \) there exist an \( \omega \) such that \( x^* \) is globally asymptotically stabilisable via the output feedback control law \( u = \omega (y - y^*) \) where \( y^* = qx_1^* \).
Proof 4.2 The control investigated is based on the Jurdjivic-Quinn stabilization procedure (see [9]). This method works well for the continuous age structured fishery system in the canonical form (16).

The candidate stabilizer is:

\[ u = - g(x) \cdot \nabla V(x) \]

where

\[ \nabla V(x) = \begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{pmatrix} \]

Using the above Lyapunov function that permits stabilize the system (3) with a constant fishing effort.

\[ V(x) = \omega_1 \left( x_1 - x_1^* - x_1^* \ln \left( \frac{x_1}{x_1^*} \right) \right) + \omega_2 \left( x_2 - x_2^* - x_2^* \ln \left( \frac{x_2}{x_2^*} \right) \right) \]

where: \( \omega_i \), \( i = 1, 2 \) are strictly positive coefficients.

It follows that:

\[ u = \begin{pmatrix} \omega_1 \left( 1 - \frac{x_1}{x_1^*} \right) \\ \omega_2 \left( 1 - \frac{x_2}{x_2^*} \right) \end{pmatrix} \cdot \begin{pmatrix} qx_1 \\ 0 \end{pmatrix} = \omega_1 qx_1 \left( 1 - \frac{x_1}{x_1^*} \right) = \omega_1 qx - \omega_1 qx_1^* = \omega_1 (y - y^*) \]

Let

\[ y^* = qx_1^* \]

The stabilizer \( u \) can easily extended to an output feedback as follows:

\[ u = \omega (y - y^*) \]

which is equivalent to

\[ E(t) = E + \omega (y - y^*) \quad (17) \]

In order to guarantee the positivity of the fishing effort we need the above condition \( y_{\text{min}} > y^* \).
5 Numerical Simulation

In this section we assign some values to the parameters of the system (3) and compute some simulations using those data. These numerical values are (see [3]): $r = \frac{7}{10}$, $s = \frac{5}{16}$, $q = \frac{25}{100}$, $E = \frac{5}{10}$, $K = 10$, $L = \frac{22}{10}$, $\sigma_1 = \frac{2}{10}$ and $\sigma_2 = \frac{1}{10}$. For these parameters, the numerical equilibrium point is $x^* = (5, 5)$ and the numerical values of the parameters $\lambda_1 = 0.2$ and $\lambda_2 = 31.8$. The parameter of regulation $\omega = 10$ in the equation (17) is chosen as belonging to the interval $[\lambda_1, \lambda_2]$. The components of the closed-loop system obtained of this fishery are illustrated in figures 1-2.

![Figure 1: Time evolution of $x_1$](image-url)
6 Conclusion

It is more desirable to use output feedback control rather than state feedback to stabilize the age structured fishery system. But the output feedback control problem for nonlinear is very difficult to solve in general, because observers are hard to construct. To avoid this problem, a linear state feedback control law is investigated and is extended to an output feedback control that permits stabilization of the continuous age structured model around the steady states. To this end, the Lyapunov function technique is adapted to the studied system. The simulation results demonstrate the effectiveness of the proposed method. One drawback of the model is structured model only of two states, therefore it would be necessary to see what will happen for a model of dimension more than two stages. This work does not claim to solve the problem of output feedback control for the continuous age structured model but it is a first step towards the determination of a complete solution.

References

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