Optimization of the Difference of Increasing and Radiant Functions

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Abstract

In this paper, we present necessary and sufficient conditions for the global minimum of the difference of non-positive strictly increasing and radiant functions. Also, a characterization of the dual problem for this class of functions is presented.

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1 Introduction

Recently many authors have discussed the theoretical development of optimality conditions for certain classes of global optimization problems (see [3, 4, 7]). One of the most important global optimization problems is that of minimizing a DC function (difference of two convex functions) that is

\[
\text{minimize} \; h(x) \; \text{subject to} \; x \in X,
\]

where \( h(x) = f(x) - g(x) \) and \( f, g \) are convex functions. In a general case, DC functions can be replaced by DAC functions (difference of two abstract convex functions)
functions), in particular, the minimizing of the difference of two increasing and co-radiant (ICR) functions and also the minimizing of the difference of two increasing and convex along rays functions (see, for example, [2, 5, 6]). In this paper, we replace \( f \) and \( g \) by non-positive increasing and radiant functions (IR functions) and we present a necessary and sufficient condition for the global minimum of \( h \). We also outline a dual approach to the study of the global optimization problem for these functions. Our approach is based on the Toland-Singer formula and some results were obtained in [1].

2 Preliminaries

Let \( X \) be a topological vector space. We assume that \( X \) is equipped with a closed convex pointed cone \( S \) (the latter means that \( S \cap (-S) = \{0\} \)). We say \( x \leq y \) or \( y \geq x \) if and only if \( y - x \in S \).

A function \( f : X \rightarrow [-\infty, +\infty] \) is called radiant if \( f(\lambda x) \leq \lambda f(x) \) for all \( x \in X \) and all \( \lambda \in (0, 1] \). It is easy to see that \( f \) is radiant if \( f(\lambda x) \geq \lambda f(x) \) for all \( x \in X \) and all \( \lambda \geq 1 \). The function \( f \) is called increasing if \( x \geq y \implies f(x) \geq f(y) \). A function \( f : X \rightarrow [-\infty, +\infty] \) is called IR if \( f \) is increasing and radiant.

Let \( X \) be a set and \( L \) be a set of real valued functions \( l : X \rightarrow \mathbb{R} \), which will be called abstract linear. For each \( l \in L \) and \( c \in \mathbb{R} \), consider the shift \( h_{l,c} \) of \( l \) on the constant \( c : h_{l,c}(x) := l(x) - c, \quad x \in X \). The function \( h_{l,c} \) is called \( L \)-affine. Recall (see [9]) that the set \( L \) is called a set of abstract linear functions if \( h_{l,c} \notin L \) for all \( l \in L \) and all \( c \in \mathbb{R} \setminus \{0\} \). The set of all \( L \)-affine functions will be denoted by \( H_L \).

A function \( f : X \rightarrow (-\infty, +\infty] \) is called proper if \( \text{dom } f \neq \emptyset \), where \( \text{dom } f \) is defined by \( \text{dom } f := \{ x \in X : f(x) < +\infty \} \). Let \( \mathcal{F}(X) \) be the union of all functions \( f : X \rightarrow (-\infty, +\infty] \) and the function \( -\infty \).

Recall (see [9]) that a function \( f \in \mathcal{F}(X) \) is called \( H \)-convex (\( H = L \), or \( H = H_L \)) if \( f(x) = \sup\{h(x) : h \in \text{supp } (f, H)\} \) for all \( x \in X \), where \( \text{supp } (f, H) := \{ h \in H : h \leq f \} \) is called the support set of the function \( f \), and \( h \leq f \) if and only if \( h(x) \leq f(x) \) for all \( x \in X \). For a function \( f \in \mathcal{F}(X) \), define the Fenchel-Moreau \( L \)-conjugate \( f^* \) of \( f \) (see [9]) by \( f^*(l) := \sup_{x \in X} [l(x) - f(x)] \), \( (l \in L) \).

Let \( f : X \rightarrow (-\infty, +\infty] \) be a function and \( x_0 \in \text{dom } f \). Recall (see [9]) that an element \( l \in L \) is called an \( L \)-subgradient of \( f \) at \( x_0 \) if \( f(x) \geq f(x_0) + l(x) - l(x_0) \) for all \( x \in X \). The set \( \partial_L f(x_0) \) of all \( L \)-subgradients of \( f \) at \( x_0 \) is called \( L \)-subdifferential of \( f \) at \( x_0 \). Now, consider the function \( u : X \times X \times (-\infty, 0) \rightarrow [-\infty, 0] \) is defined by:

\[
u(x, y, \beta) := \sup\{ \lambda \leq \beta : \lambda y \geq -x\}, \quad (x, y \in X; \beta \in (-\infty, 0)), \quad (2.1)\]

(we use the convention \( \sup \emptyset = -\infty \)).
The function $u$ was introduced and examined in [8]. In the sequel, for each $y \in X$ and each $\beta < 0$, we consider the function $u_{(y,\beta)} : X \to [-\infty, 0]$ is defined by $u_{(y,\beta)}(x) := u(x, y, \beta)$ for all $x \in X$, and set $L := \{u_{(y,\beta)} : y \in X, \beta < 0\}$. It is easy to check that $L$ is a set of IR functions. In the following, we gather some results for non-positive IR functions which will be used later.

**Proposition 2.1.** ([8], Proposition 4.1) Let $f : X \to [-\infty, 0]$ be an IR function. Then

$$\text{supp } (f, L) = \{u_{(y,\beta)} \in L : f(-\beta y) \geq \beta\}. \quad (2.2)$$

**Theorem 2.1.** ([8], Theorem 4.1) Let $f : X \to [-\infty, 0]$ be an IR function and $x_0 \in X$ be such that $f(x_0) \neq -\infty$. Then

$$D := \{u_{(y,\beta)} \in L : f(x_0) \geq u_{(y,\beta)}(x_0), \beta - u_{(y,\beta)}(x_0) \leq f(-\beta y) - f(x_0)\} \subset \partial_L f(x_0). \quad (2.3)$$

Moreover, the equality holds if and only if $f(x_0) = 0$.

# 3 Dual Optimality Conditions

Let $f, g : X \to [-\infty, +\infty]$ be proper functions. Let $h := f - g$. Now, consider the following extremal problem:

$$h(x) \to \text{min} \quad \text{subject to } x \in X. \quad (3.1)$$

Clearly, if $\inf_{x \in X} h(x) = -\infty$, then (3.1) is trivial. Therefore, we consider $\inf_{x \in X} h(x) > -\infty$. Now, consider the following problem:

$$g^*(l) - f^*(l) \to \text{min} \quad \text{subject to } l \in \text{dom } f^*. \quad (3.2)$$

The problem defined by (3.2) is called the dual problem with respect to (3.1). In the following, we give the well-known Toland-Singer formula (see [10, 11]).

**Theorem 3.1.** Let $Z$ be a set and $U$ be a set of real valued abstract linear functions defined on $Z$. Let $f, g : Z \to (-\infty, +\infty]$ be proper $H_U$-convex functions such that $\inf_{x \in Z} [f(x) - g(x)] > -\infty$. Then, $\inf \{f(x) - g(x) : x \in Z\} = \inf \{g^*(l) - f^*(l) : l \in U\}$.

So the following result can be obtained directly from Theorem 3.1.

**Proposition 3.1.** Let $f, g : X \to [-\infty, 0]$ be proper IR functions such that $\inf_{x \in X} [f(x) - g(x)] > -\infty$. Then, $\inf \{f(x) - g(x) : x \in X\} = \inf \{g^*(u_{(y,\beta)}) - f^*(u_{(y,\beta)}) : u_{(y,\beta)} \in L\}$. 

Lemma 3.1. Let \( f, g : X \to [-\infty, 0] \) be IR functions, \( x \in X \) be such that \( f(x) + g(x) > -\infty \), and let \( \epsilon < \min\{f(x), g(x)\} \) be arbitrary. Then, \( u_{\epsilon}(x) \in \partial f(x) \cap \partial g(x) \). Moreover, \( g^*(u_{\epsilon}(x)) - f^*(u_{\epsilon}(x)) = f(x) - g(x) \).

Proof: Since \( f(x) > \epsilon \) and by the definition of \( u_{\epsilon}(x) \) one has \( u_{\epsilon}(x) = \epsilon \), then in view of Theorem 2.1 we obtain \( u_{\epsilon}(x) \in \partial f(x) \). By a similar argument we have \( u_{\epsilon}(x) \in \partial g(x) \). Now, since \( u_{\epsilon}(x) \in \partial f(x) \), then by Fenchel-Young equality we have \( f^*(u_{\epsilon}(x)) = u_{\epsilon}(x) - f(x) \). Therefore, we conclude that

\[
\begin{aligned}
g^*(u_{\epsilon}(x)) - f^*(u_{\epsilon}(x)) &= u_{\epsilon}(x) - g(x) - u_{\epsilon}(x) + f(x) \\
&= f(x) - g(x),
\end{aligned}
\]

which completes the proof.

Theorem 3.2. Let \( f, g : X \to [-\infty, 0] \) be proper IR functions such that \( \inf_{x \in X} [f(x) - g(x)] > -\infty \). Let \( x_0 \in X \) and \( \epsilon < \min\{f(x_0), g(x_0)\} \). Then, \( x_0 \) is a global minimizer of the problem (3.1) if and only if \( u_{\epsilon}(x_0) \) is a global minimizer of the problem (3.2).

Proof: Suppose that \( x_0 \) is a global minimizer of the problem (3.1). Since \( \epsilon < \min\{f(x_0), g(x_0)\} \), then, by Lemma 3.1 and Proposition 3.1 we get

\[
\begin{aligned}
g^*(u_{\epsilon}(x_0)) - f^*(u_{\epsilon}(x_0)) &= f(x_0) - g(x_0) \\
&= \inf_{x \in X} [f(x) - g(x)] \\
&= \inf \{g^*(u_{\epsilon}(x)) - f^*(u_{\epsilon}(x)) : u_{\epsilon}(x) \in L \}.
\end{aligned}
\]

Hence \( u_{\epsilon}(x_0) \) is a global minimizer of the problem (3.2). Conversely, assume that \( u_{\epsilon}(x_0) \) is a global minimizer of the problem (3.2). Then, it follows from Lemma 3.1 and Proposition 3.1 that \( x_0 \) is a global minimizer of the problem (3.1).

4 Necessary and Sufficient Conditions

In this section, we present necessary and sufficient conditions for the global minimum of the difference of strictly non-positive IR functions. First, consider the function \( h := g - f \), where \( f, g : X \to [-\infty, +\infty] \) are proper functions. Let \( \eta := \inf_{x \in X} h(x) > -\infty \). This implies that \( f(x) \leq g(x) - \eta, \forall x \in X \). Let \( \tilde{g}(x) := g(x) - \eta \). It is easy to see that \( f(x) \leq \tilde{g}(x) \) for all \( x \in X \) if and only if \( \text{supp } (f, L) \subset \text{supp } (\tilde{g}, L) \), or equivalently, \( x_0 \) is a global minimizer of the function \( h \) if and only if

\[
\text{supp } (f, L) \subset \text{supp } (\tilde{g}, L).
\] (4.1)
Now, consider a set $U$ of functions defined on a set $Z$. We assume that $U$ is equipped with the natural (pointwise) order relation. Recall (see [9]) that a function $f$ is called a maximal element of the set $U$, if $f \in U$ and $\bar{f} \in U$, $f(x) \geq \bar{f}(x)$ for all $x \in Z$ $\implies \bar{f} = f$. We now concentrate on the support set of IR functions and we obtain some results which will be used later.

**Proposition 4.1.** Let $f : X \to [-\infty, 0]$ be an IR function and let $u_{(y,\beta)} \in \text{supp } (f, L)$. Assume that $u_{(y,\beta)}$ is a maximal element of $\text{supp } (f, L)$. Then, $f(-\beta y) = \beta$.

**Proof:** Let $u_{(y,\beta)} \in \text{supp } (f, L)$. Then, by Proposition 2.1, we have $f(-\beta y) \geq \beta$. Consider $u_{(-\beta y, f(-\beta y))} \in L$. Then, in view of the definition of $u_{(y,\beta)}$ we conclude that $u_{(-\beta y, f(-\beta y))}(-\beta y) = f(-\beta y)$. Since $f(-\beta y) \geq \beta$, it follows from Proposition 2.1 that $u_{(-\beta y, f(-\beta y))} \in \text{supp } (f, L)$. Also, by using $f(-\beta y) \geq \beta$ and the definition of $u_{(y,\beta)}$ one has

$$u_{(y,\beta)}(x) \leq u_{(-\beta y, f(-\beta y))}(x), \ \forall x \in X. \tag{4.2}$$

Since $u_{(y,\beta)}$ is a maximal element of $\text{supp } (f, L)$, then by (4.2) we obtain

$$u_{(y,\beta)}(x) = u_{(-\beta y, f(-\beta y))}(x), \ \forall x \in X. \tag{4.3}$$

Put $x := -\beta y$ in (4.3), we get $f(-\beta y) = \beta$. 

**Remark 4.1.** The converse statement of Proposition 4.1 is not valid. Consider IR function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ for all $x \in \mathbb{R}$. It follows from Proposition 2.1 that $u_{(-1,\beta)} \in \text{supp } (f, L)$ and $f(\beta) = \beta$ for all $\beta < 0$. But the maximal element of the support set of $f$ does not exist.

In the following, we show by extra conditions that the converse statement of Proposition 4.1 holds. The proof of the following result is similar to the one of Proposition 4.2 in [2], and so we omit it.

**Proposition 4.2.** Let $f : X \to [-\infty, 0]$ be a strictly IR function. Let $y \in X$ be such that $\varepsilon_y := \max\{\beta < 0 : f(-\beta y) \geq \beta\} < 0$. Then, $u_{(y,\varepsilon_y)}$ is a maximal element of the support set of $f$ if and only if $f(-\varepsilon_y y) = \varepsilon_y$.

**Corollary 4.1.** Let $f : X \to [-\infty, 0]$ be a strictly IR function such that $\varepsilon_y := \max\{\beta < 0 : f(-\beta y) \geq \beta\} < 0$ ($y \in X$). Then for each $u_{(y,\beta)} \in \text{supp } (f, L)$ there exists a maximal element $u_{(\tilde{y},\tilde{\beta})}$ of the support set of $f$ such that $u_{(y,\beta)} \leq u_{(\tilde{y},\tilde{\beta})}$. In this case, we have $\tilde{y} = \frac{-\varepsilon_y y}{-f(-\varepsilon_y y)}$, and $\tilde{\beta} = f(-\varepsilon_y y)$.

**Proof:** This is an immediate consequence of Proposition 4.1 and Proposition 4.2.
Theorem 4.1. Let \( f, g : X \to [-\infty, 0] \) be strictly IR functions such that 
\[ \varepsilon_y := \max\{\beta < 0 : f(\beta y) \geq \beta\} < 0, \] 
and \( \eta_z := \max\{\beta < 0 : g(\beta z) \geq \beta\} < 0 \) (\( y, z \in X \)). Then the following assertions are equivalent:

(i) \( \text{supp}(f,L) \subset \text{supp}(g,L) \).

(ii) For each maximal element \( u_{(y,\varepsilon_y)} \) of \( \text{supp}(f,L) \) there exists a maximal element \( u_{(z,\eta_z)} \) of \( \text{supp}(g,L) \) such that \( u_{(y,\varepsilon_y)}(x) \leq u_{(z,\eta_z)}(x) \) for all \( x \in X \).

Proof: (i) \( \Rightarrow \) (ii). Suppose that \( \text{supp}(f,L) \subset \text{supp}(g,L) \). Let \( u_{(y,\varepsilon_y)} \) be a maximal element of \( \text{supp}(f,L) \), then \( u_{(y,\varepsilon_y)} \in \text{supp}(g,L) \). Thus, by Corollary 4.1 there exists a maximal element \( u_{(z,\eta_z)} \) of \( \text{supp}(g,L) \) such that \( u_{(y,\varepsilon_y)}(x) \leq u_{(z,\eta_z)}(x) \) for all \( x \in X \).

(ii) \( \Rightarrow \) (i). Let \( u \in \text{supp}(f,L) \) be arbitrary. Then by Corollary 4.1 there exists a maximal element \( u_{(y,\varepsilon_y)} \) of \( \text{supp}(f,L) \) such that \( u \leq u_{(y,\varepsilon_y)} \). Therefore, by the hypothesis (ii) there exists a maximal element \( u_{(z,\eta_z)} \in \text{supp}(g,L) \) such that \( u_{(y,\varepsilon_y)} \leq u_{(z,\eta_z)} \). Then, \( u_{(z,\eta_z)} \geq u \), and hence \( u \in \text{supp}(g,L) \). This completes the proof.

In the following, we present necessary and sufficient conditions for the global minimum of the difference of strictly IR functions.

Theorem 4.2. Let \( f, g : X \to [-\infty, 0] \) be strictly IR functions such that \( f(x) \leq g(x) \) for all \( x \in X \). Then, \( x_0 \in X \) is a global minimizer of the function \( h := g - f \) if and only if for each \( y \in X \) with \( f(-\varepsilon_y y) = \varepsilon_y \) there exists \( z \in X \) with \( \tilde{g}(-\eta_z z) = \eta_z \) such that \( u_{(y,\varepsilon_y)} \leq u_{(z,\eta_z)} \), where \( \tilde{g}(x) := g(x) - h(x_0) \) for all \( x \in X \), \( \varepsilon_y := \max\{\beta < 0 : f(\beta y) \geq \beta\} < 0 \) and \( \eta_z := \max\{\beta < 0 : g(\beta z) \geq \beta\} < 0 \) (\( y, z \in X \)). It is worth noting that since \( h(x_0) \geq 0 \), then one has \( \tilde{g} \) is a strictly IR function.

Proof: Due to (4.1), \( x_0 \) is a global minimizer of the function \( h \) if and only if \( \text{supp}(f,L) \subset \text{supp}(\tilde{g},L) \). Now, the result follows from Proposition 4.2 and Theorem 4.1.

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