Delay Differential AK Model
with Non-positive Population Growth Rate

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Abstract

This paper develops an AK Solow-Swan model with non-positive population growth rate and production lag. The resulting model happens to have an unbounded positive solution and all bounded solutions that are oscillatory.

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1 Introduction

Differential equations with time delay play an important role in many fields such as economy, engineering, biology and social sciences. Recently, some authors have explored the consequences of looking into neoclassical models under non-constant population growth laws (see, e.g., Ferrara and Guerrini [1-3], Guerrini [4-9]). In this paper, we consider the economical model of Solow-Swan with AK technology and non-positive population growth rate, introduced by Ferrara [1], assuming that new capital is not produced instantaneously, but it is produced and installed after $T > 0$ periods. We find that augmenting the model with a production lag changes its behaviour. There is an unbounded positive solution and every bounded solution oscillates, whereas Ferrara’s model [1] has a unique solution that increases monotonically to infinity.

2 The model

Let us start considering the Solow-Swan model with AK technology and a non-positive rate of population growth (see Ferrara [1]), whose structure in
continuous time is given by $\dot{k}_t = (sA - n_t)k_t$, where $k_t$ is per capita capital stock, $s \in (0, 1)$ is the saving rate and $A > 0$ is a parameter reflecting the level of the technology. For simplicity, there is no depreciation of capital. Population $L_t$ is assumed to have a growth rate $n_t = \frac{L_t}{L_0} - 1$ that is bounded, $-m \leq n_t \leq 0$ ($m > 0$) and such that there exists $\lim_{t \to \infty} n_t = n_\infty$. Notice first that it must be $n_\infty = 0$. In fact, if today’s population is normalized to one, $L_0 = 1$, integrating the previous inequality between 0, $t$, and, then, exponentiating the result, we get $e^{-mt} \leq L_t \leq 1$. But $L_t$ is monotone decreasing yields $L_\infty = \lim_{t \to \infty} L_t$, so that $0 \leq L_\infty \leq 1$. Now, applied Lemma 1 of Guerrini [5]. Next, we suppose that there is a delay of $T > 0$ period before capital can be used for production. At time $t$, the productive capital is $k_{t-T}$. In this way, the capital market equilibrium condition in this model becomes

$$\dot{k}_t = (sA - n_t)k_{t-T}, \quad (1)$$

for some initial function $k_t = \phi_t$, $t \in [-T, 0]$. Instead of an initial point value for an ordinary differential equation, the initial function $\phi_t$ is required, which is defined over the range of time delimited by the delay.

3 Behaviour of solutions

We now investigate the first-order linear delay differential equation (1).

**Proposition 3.1.** Eq. (1) has an unbounded positive solution and every bounded solution of (1) has arbitrarily large zeros, i.e. it is oscillatory.

**Proof.** By a simple calculation, one obtain $0 < sA \leq sA - n_t \leq sA + m$ from being $-m \leq n_t \leq 0$. Setting $p_t = sA - n_t$, we see that this function is positive and satisfies $\int_0^{\infty} p_t = \infty$. Consequently, an application of Theorem 2.1. from Lalli and Zhang [10]) yields the first part of the statement. It remains to prove that every bounded solution of (1) is oscillatory. Assume the contrary and let $x_t$ be a positive bounded solution of (1). Then $x_t$ is nondecreasing and bounded, so that there exists $\lim_{t \to \infty} x_t = x_\infty > 0$. From Eq. (1), we get $\dot{x}_t \geq (sA - n_t)(x_\infty/2)$ for large $t$. In view of our assumption, we can conclude that $\lim_{t \to \infty} x_t = +\infty$, which contradicts the boundedness of $x_t$. \hfill \Box

**Remark 3.2.** Changes in the qualitative behaviour of the solution may be observed as a consequence of a delay term. In Ferrara [1], an economy that starts from a stock of per capita capital will perpetually accumulate physical capital and its per capita capital stock will rise toward infinity. On the contrary, the presence of a deviating argument causes the oscillation of solutions.
References


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