Approximate Analytical Solutions of Two Dimensional Transient Heat Conduction Equations

M. Mahalakshmi
Department of Mathematics
School of Humanities & Sciences
SASTRA University
Thanjavur-613 401, Tamilnadu, India

R. Rajaraman
Department of Mathematics
School of Humanities & Sciences
SASTRA University
Thanjavur-613 401, Tamilnadu, India

G. Hariharan
Department of Mathematics
School of Humanities & Sciences
SASTRA University
Thanjavur-613 401, Tamilnadu, India
hariharan@maths.sastra.edu

K. Kannan
Department of Mathematics
School of Humanities & Sciences
SASTRA University
Thanjavur-613 401, Tamilnadu, India

Abstract
In this paper, the Homotopy analysis method (HAM) is employed to obtain the analytical and approximate solutions of the two-dimensional heat conduction equations. Some illustrative examples are presented. The solution is simple yet highly accurate and compare favorably with the solutions obtained early in the literature.
Keywords: Two-dimensional diffusion equation; Homotopy analysis method

1 Introduction

The diffusion equation arises naturally in many engineering and science applications, such as heat transfer, fluid flows, solute transports, chemical and biological processes. Consider the two-dimensional transient heat equations

\[
\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}
\]

with the initial conditions \( U(x, t, 0) = f(x, y, t) \). Many practical applications are transient in nature and in such problems the temperature varies with respect to time. For example, in a lot of components of industrial plants such as boilers, refrigeration and air-conditioning equipment, the heat transfer process is transient during the initial stages of operation. Therefore, the analysis of transient heat conduction is very important. Conventional numerical techniques such as finite difference method (FDM), finite element method (FEM), finite volume method (FVM) are used to solve complicated problems. The smoothed partial hydrodynamic (SPA) method [1], meshless method [2], Local RBF-DQ method [3] established for solving the two-dimensional heat flow equation. Akbarzade et al. [4] introduced the two and three-dimensional diffusion problems by HPM and VIM. Very recently, some approximate analytical solutions are established such as Exp-function method [5], Variational iteration method (VIM) [6], Homotopy perturbation method (HPM) [7-9] and Energy Balance method (HBM) [6].

The widely applied methods (i.e., perturbation technique) are of great interest to be used in engineering and science. In 1992, Liao [10-17] established the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely, homotopy analysis method (HAM). It provides us with a simple way to adjust and control the convergence of solution series. This method has been successfully applied to solve many types of linear and nonlinear partial differential equations (PDEs). Abbasbandy [18] established the application of Homotopy analysis method to nonlinear equations arising in heat transfer. After this, many types of nonlinear problems were solved with HAM by other researchers [19-24]. Applications of two-dimensional heat equation are presented in [25].
Prashant et al. [26] worked out an exact analytical solution for two-dimensional, unsteady, multilayer heat conduction in spherical coordinates.

In this paper, we use homotopy analysis method (HAM) to solve 2D heat conduction equations. To show the efficiency of the method, five problems are solved.

The paper is organized as follows. In section 2 the HAM is briefly reviewed. Section 3 deals with solving the two-dimensional heat conduction equation using HAM. In section 4, we have presented some examples. Section 5 is dedicated to conclusion.

2 Basic idea of Homotopy Analysis method (HAM):

In this section the basic ideas of the homotopy analysis method are introduced. Here a description of the method is given to handle the general nonlinear problem.

\[ Nu_0(t) = 0, \quad t > 0 \]  

Where \( N \) is a nonlinear operator and \( u_0(t) \) is unknown function of the independent variable \( t \).

2.1. Zero-order deformation equation

Let \( u_0(t) \) denote the initial guess of the exact solution of Eq. (1), \( h \neq 0 \) an auxiliary parameter, \( H(t) \neq 0 \) an auxiliary function and \( L \) is an auxiliary linear operator with the property:

\[ L(f(t)) = 0, \quad f(t) = 0. \]  

The auxiliary parameter \( h \), the auxiliary function \( H(t) \), and the auxiliary linear operator \( L \) play an important role within the HAM to adjust and control the convergence region of solution series. Liao [10] constructs, using \( q \in [0, 1] \) as an embedding parameter, the so-called zero-order deformation equation.

\[ (1 - q)L[\theta(t; q) - u_0(t)] = qH(t)N[\theta(t; q)]. \]  

where \( \theta(t; q) \) is the solution which depends on \( h, H(t), L, u_0(t) \) and \( q \). When \( q=0 \), the zero-order deformation Eq. (4) becomes

\[ \theta(t; 0) = u_0(t). \]  

And when \( q=1 \), since \( h \neq 0 \) and \( H(t) \neq 0 \), the zero-order deformation Eq.(1) reduces to,
\[ N[\mathcal{O}(t;1)] = 0, \]  
(6)

So, \( \mathcal{O}(t;1) \) is exactly the solution of the nonlinear Eq. (2). Define the so-called \( m^{th} \) order deformation derivatives.

\[ u_m(t) = \frac{1}{m!} \frac{\partial^m \mathcal{O}(t;q)}{\partial q^m} \]  
(7)

If the power series (7) of \( \mathcal{O}(t;q) \) converges at \( q=1 \), then we get the following series solution:

\[ u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t). \]  
(8)

where the terms \( u_m(t) \) can be determined by the so-called high order deformation described below.

2.2. High-order deformation equation

Define the vector,

\[ \overrightarrow{u_n} = [u_0(t), u_1(t), u_2(t), \ldots, u_n(t)] \]  
(9)

Differentiating Eq.(4) \( m \) times with respect to embedding parameter \( q \), the setting \( q=0 \) and dividing them by \( m! \), we have the so-called \( m^{th} \) order deformation equation.

\[ L[u_m(t) \mathcal{R}_m u_{m-1}(t)] = hH(t)R_m \overrightarrow{u_m}(t), \]  
(10)

where

\[ \mathcal{R}_m = \begin{cases} 0, & m \leq 1 \\ 1, & \text{otherwise} \end{cases} \]  
(11)

and

\[ R_m \overrightarrow{u_m}(t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\mathcal{O}(t;q)]}{\partial q^{m-1}} \]  
(12)

For any given nonlinear operator \( N \), the term \( R_m \overrightarrow{u_m}(t) \) can be easily expressed by (12). Thus, we can gain \( u_1(t), u_2(t), \ldots \) by means of solving the linear high-order deformation Eq. (10) one after the other in order. The \( m^{th} \)–order approximation of \( u(t) \) is given by
Approximate analytical solutions

\[ u(t) = \sum_{k=0}^{m} u_k(t) \]  \hspace{1cm} (13)

ADM, VIM and HPM are special cases of HAM when we set \( h = -1 \) and \( H(r,t) = 1 \) in Eq. (10).

We will get the same solutions for all the problems by above methods when we set \( h = -1 \) and \( H(r,t) = 1 \). When the base functions are introduced the \( H(r,t) = 1 \) is properly chosen using the rule of solution expression, rule of coefficient of ergodicity and rule of solution existence.

3 Method of solution by Homotopy analysis method (HAM)

Consider the equation

\[ u_t = u_{xx} + u_{yy} \]  \hspace{1cm} (14)

With initial condition

\[ u(x, y, 0) = e^{x+y} \]  \hspace{1cm} (15)

We apply Homotopy analysis method (HAM) to Eq. (14) and (15), as follows:

Since \( m \geq 1 \), \( \chi_m = 1 \). Set \( h = -1 \) and \( H(r,t) = 1 \) in Eq. (10). Then Eq. (10) becomes

\[ u_m(x, y, t) = u_{m-1}(x, y, 0) - L^{-1}(R_m(u_{m-1}, x, y, t)) \]  \hspace{1cm} (16)

where

\[ R_m(u_{m-1}, x, y, t) = \frac{\partial^2 u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{\partial^2 u_{m-1}}{\partial y^2} \]  \hspace{1cm} (17)

In the same way, we obtain \( u_1(x, y, t), u_2(x, y, t), u_3(x, y, t) \) as:

\[ u_1(x, y, t) = 2te^{x+y} \],
\[ u_2(x, y, t) = 2t^2 e^{x+y} \],
\[ u_3(x, y, t) = \frac{4}{3} t^2 e^{x+y} \]
The exact solution in a closed form is given by

\[ u(x, y, t) = e^{(2t + x + y)} \]  

(18)

4. Test Problems

**Example 1.** Consider the Eq. (14) with the initial condition

\[ u(x, y, 0) = \sin x \cos y \]

By HAM,

\[ u_0 = \sin x \cos y, \]
\[ u_1 = -2t \sin x \cos y, \]
\[ u_2 = \frac{4t^2}{2} \sin x \cos y, \]
\[ u_3 = -\frac{8t^3}{6} \sin x \cos y \]

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Then the exact solution in a closed form is

\[ u(x, y, t) = \sin x \cos y e^{-2t} \]  

(19)

**Example 2.**

Consider the Eq. (14) with the initial condition
\[ u(x, y, 0) = (1 - y)e^x \]

By HAM,
\[ u_0 = (1 - y)e^x, \]
\[ u_1 = t(1 - y)e^x, \]
\[ u_2 = \frac{t^2}{2}(1 - y)e^x, \]
\[ u_3 = \frac{t^3}{6}(1 - y)e^x \]
\[ u_4 = \frac{t^4}{24}(1 - y)e^x \]

Then the exact solution in a closed form is
\[ u(x, y, t) = (1 - y)e^{x + t}. \] (20)

Example 3.

Consider the Eq. (14) with the initial condition
\[ u(x, y, 0) = \sinh x \sinh y \]

By HAM,
\[ u_0 = \sinh x \sinh y \]
\[ u_1 = 2t \sinh x \sinh y, \]
\[ u_2 = \frac{4t^2}{2} \sinh x \sinh y, \]
\[ u_3 = \frac{8t^3}{6} \sinh x \sinh y, \]
\[ u_4 = \frac{16t^4}{24} \sinh x \sinh y \]
Then the exact solution in a closed form is

\[ u(x, y, t) = \sinh x \sinh ye^{2t} \]  \hspace{1cm} (21)

**Example 4.**

We consider the Eq. (14) with the initial condition

\[ u(x, y, 0) = e^{x-y} \].

Then

\[ u_0 = e^{x-y}, \]
\[ u_1 = 2te^{x-y}, \]
\[ u_2 = \frac{4t^2}{2} e^{x-y}, \]
\[ u_3 = \frac{8t^3}{6} e^{x-y}, \]
\[ u_4 = \frac{15t^4}{24} e^{x-y} \]

The exact solution in a closed form is given by

\[ u(x, y, t) = e^{x-y} + 2t \]  \hspace{1cm} (22)

**Example 5.**

We consider the Eq. (14) with the initial condition

\[ u(x, y, 0) = (1 - x)e^y \]

By HAM,

\[ u_0 = (1 - x)e^y, \quad u_1 = t(1 - x)e^y, \quad u_2 = \frac{t^2}{2} (1 - x)e^y, \]
\[ u_3 = \frac{t^3}{6} (1 - x) e^y, \]
\[ u_4 = \frac{t^4}{24} (1 - x) e^y. \]

Then the exact solution in a closed form is

\[ u(x, y, t) = (1 - x) e^{y+t} \] (23)

5. Concluding Remarks

In this paper, the Homotopy analysis method has been successfully applied to obtain the analytical/approximate solutions of the two-dimensional unsteady state heat conduction equations. The results show that HAM is very effective and it is a convenient tool to solve the two-dimensional heat conduction problem. This method gives us a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. In conclusion, it provides accurate exact solution for linear problems.

References


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