Some New Results in Dislocated and Dislocated Quasi-Metric Spaces

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Abstract
In this paper we have proved two fixed point theorems in complete dislocated and dislocated quasi-metric spaces, which generalizes some recent results in literature.

Keywords: Dislocated quasi-metric space; fixed point; dq-limit; dq-Cauchy sequence
1 Introduction

Hitzler and Seda, introduced the notion of dislocated metric spaces [5, 6] and generalized the Banach Contraction Principle in such spaces. These metrics play a very important role not only in topology but also in other branches of science involving mathematics especially in logic programming and electronic engineering. D. S. Jaggi [3] proved fixed point theorem using rational type of contractive condition which generalized the Banach contraction principle in complete metric space. Zeyada et al. [4] initiated the concept of dislocated quasi-metric space and generalized the result of Hitzler and Seda in dislocated quasi-metric spaces. Results on fixed points in dislocated and dislocated quasi-metric spaces followed by Isufati [1] and Aage and Salunke [3], and recently by Shrivastava, Ansari and Sharma [7].

In this paper we establish two fixed point theorems in the context of dislocated quasi-metric space, which generalize and unify some known results.

2 Preliminaries

We introduce below necessary notions and present a few results in dislocated quasi-metric space, that will be used throughout the paper.

**Definition 2.1** [4] Let \( X \) be a non-empty and let \( d : X \times X \to \mathbb{R}^+ \) be a function, called a distance function if for all \( x, y, z \in X \), satisfies:

\[
\begin{align*}
   &d_1 : d(x, x) = 0 \\
   &d_2 : d(x, y) = d(y, x) = 0 \Rightarrow x = y \\
   &d_3 : d(x, y) = d(y, x) \\
   &d_4 : d(x, y) \leq d(x, z) + d(z, y).
\end{align*}
\]

If \( d \) satisfies the condition \( d_1 - d_4 \), then \( d \) is called a metric on \( X \). If it satisfies the conditions \( d_1, d_2 \) and \( d_4 \) it is called a quasi-metric space. If \( d \) satisfies conditions \( d_2, d_3 \) and \( d_4 \) it is called a dislocated metric (or simply \( d \)-metric). If \( d \) satisfies only \( d_2 \) and \( d_4 \) then \( d \) is called a dislocated quasi-metric (or simply \( dq \)-metric) on \( X \). A nonempty set \( X \) with \( dq \)-metric \( d \), i. e., \((X, d)\) is called a dislocated quasi-metric space.

**Definition 2.2** [4] A sequence \( (x_n)_{n \in \mathbb{N}} \) in \( dq \)-metric space \((X, d)\) is called Cauchy if for all \( \varepsilon > 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( \forall m, n \geq n_0 \), \( d(x_m, x_n) < \varepsilon \) or \( d(x_n, x_m) < \varepsilon \).

In above definition if we replace \( d(x_m, x_n) < \varepsilon \) or \( d(x_n, x_m) < \varepsilon \) by \( \max\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon \) then \( (x_n)_{n \in \mathbb{N}} \) is called "bi" Cauchy sequence.
Definition 2.3 [4] A sequence \((x_n)_{n \in \mathbb{N}}\) dislocated quasi-converges or dq-converges to \(x\) if \(\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0\).

In this case \(x\) in called a dq-limit of \((x_n)_{n \in \mathbb{N}}\) and we write \(x_n \to x\).

Proposition 2.4[4] Every convergent sequence is a dq-metric space is "bi" Cauchy.

Definition 2.5 [4] A dq-metric space \((X, d)\) is complete if every Cauchy sequence in it is dq-convergent.

Lemma 2.6 [4] Every subsequence of dq-convergent sequence to a point \(x_0\) is dq-convergent to \(x_0\).

Definition 2.7 [4] Let \((X, d)\) be a dq-metric space. A mapping \(f : X \to X\) is called contraction if there exists \(0 < \lambda < 1\) such that:

\[d(fx, fy) \leq \lambda d(x, y)\]

for all \(x, y \in X\).

Lemma 2.8 [4] Let \((X, d)\) be a dq-metric space. If \(f : X \to X\) is a contraction function, then \(f^n(x_0)\) is a Cauchy sequence for each \(x_0 \in X\).

Lemma 2.9 [4] dq-limits in a dq-metric space are unique.

Further the following theorems give common fixed points for continuous contraction mappings satisfying contractive type conditions and rational inequality in dislocated and dislocated quasi-metric space. Our theorems unify and generalizes some results.

Theorem 2.10 [4] Let \((X, d)\) be complete dq-metric space and let \(f : X \to X\) be a continuous contraction function then \(f\) has a unique fixed point.

Theorem 2.11 [3] Let \(T\) be a continuous self-map defined on a complete metric space \((X, d)\). Further, let \(T\) satisfies the following contractive condition:

\[d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y)\]  \(1\)

for all \(x, y \in X\), \(x \neq y\) and for some \(\alpha, \beta \in [0,1]\) with \(\alpha + \beta < 1\), then \(T\) has a unique fixed point.

Theorem 2.12 [7] Let \(T\) be a continuous self mapping defined on a complete dq-metric space \((X, d)\). Further let \(T\) satisfy the contractive condition \((1)\), then \(T\) has a unique fixed point.

Theorem 2.13 [7] Let \((X, d)\) be a complete dislocated quasi-metric space. Let \(T : X \to X\) be continuous mapping satisfies the condition:

\[d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \gamma [d(x, Tx) + d(y, Ty)] + \delta [d(x, Ty) + d(y, Tx)]\]  \(2\)

for all \(x, y \in X\), \(\alpha, \beta, \gamma, \delta \in [0,1]\) and \(0 \leq \alpha + \beta + 2\gamma + 2\delta < 1\). Then \(T\) has a unique fixed point.
3 Main results

Our result is the following theorem which unifies the results of [1], [2] and [7].

**Theorem 3.1** Let \((X, d)\) be a complete dq-metric space. Let \(T : X \to X\) be continuous mapping satisfies the condition:

\[
\begin{align*}
    d(Tx, Ty) &\leq \alpha d(x, y) + \beta \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \gamma [d(x, Tx) + d(y, Ty)] + \\
    &\quad + \delta [d(x, Ty) + d(y, Tx)] + \eta [d(x, Tx) + d(x, y)]
\end{align*}
\]

for all \(x, y \in X\), \(x, y \in X\), \(\alpha, \beta, \gamma, \delta, \eta\) non negative with \(0 \leq \alpha + \beta + 2\gamma + 2\delta + 2\eta < 1\), then \(T\) has a unique fixed point.

**Proof:** Let be any \(x_0 \in X\) and define the sequence as follows:

\(T(x_0) = x_1, T(x_1) = x_2, \ldots, T(x_n) = x_{n+1}, \ldots\)

Putting \(x = x_{n+1}\) and \(y = x_n\) in (3) we have:

\[
d(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1})
\]

\[
\leq \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \gamma [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + \\
\quad + \delta [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] + \eta [d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)]
\]

\[
= \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \gamma [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \\
\quad + \delta d(x_{n-1}, x_n) + \eta d(x_{n-1}, x_n)
\]

\[
(\alpha + \gamma + \delta + 2\eta) d(x_{n-1}, x_n) + (\beta + \gamma + \delta) d(x_n, x_{n+1})
\]

Therefore:

\[
d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)
\]

where \(\lambda = \frac{\alpha + \gamma + \delta + 2\eta}{1 - (\beta + \gamma + \delta)}\), \(0 \leq \lambda < 1\).

So

\[
d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)
\]

Similarly \(d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})\) . From (4) we have

\[
d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1})
\]

Continuing in this way, we have \(d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)\).

Since \(0 \leq \lambda < 1\), for \(n \to \infty\), we have \(d(x_n, x_{n+1}) \to 0\).
Similarly, we show that \( d(x_{n+1}, x_n) \to 0 \). Hence \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in complete dislocated quasi-metric space \((X, d)\). So there exists \( u \in X \) such that 
\[ (x_n)_{n \in \mathbb{N}} \text{ dislocated quasi converges to } u. \]
Since \( T \) is a continuous, therefore 
\[ T(u) = T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} (x_{n+1}) = u. \]
Thus, \( u \) is a fixed point of \( T \).

**Uniqueness:** Suppose \( u \) and \( v \) are two fixed points of \( T \) \( (u \neq v, Tu = u, Tv = v) \).
Let \( u \) be a fixed. Then by condition [3] for \( u \) we have:
\[
d(u, u) = d(Tu, Tu) \\
\leq \alpha d(u, u) + \beta d(u, u) + 2\gamma d(u, u) + 2\delta d(u, u) + 2\eta d(u, u) \\
= (\alpha + \beta + 2\gamma + 2\delta + 2\eta)d(u, u)
\]
which implies that \( d(u, u) = 0 \), since \( 0 < \alpha + \beta + 2\gamma + 2\delta + 2\eta < 1 \). Thus, \( d(u, u) = 0 \) for a fixed point \( u \) of \( T \). Similarly, we get \( d(v, v) = 0 \) for \( v \) fixed point of \( T \).

Now from (3) we have
\[
d(u, v) = d(Tu, Tv) \\
\leq \alpha d(u, v) + \beta \frac{d(u, u)d(v, v)}{d(u, v)} + \gamma [d(u, u) + d(v, v)] \\
\delta [d(u, v) + d(v, u)] + \eta [d(u, u) + d(v, v)] \\
= (\alpha + \delta + \eta)d(u, v) + \delta d(v, u)
\]
Similarly:
\[
d(v, u) \leq (\alpha + \delta + \eta)d(v, u) + \delta d(u, v)
\]
Hence  \[ |d(u, v) - d(v, u)| \leq (\alpha + \eta)|d(u, v) - d(v, u)|. \] Since \( 0 < \alpha + \eta < 1 \), we get:
\[
d(u, v) = d(v, u) \quad (5)
\]
Again replacing (5) in (3) we have that \( d(u, v) \leq (\alpha + 2\delta + \eta)d(u, v) \), which gives \( d(u, v) = 0 \). Since \( 0 \leq (\alpha + 2\delta + \eta) < 1 \). Further, \( d(u, v) = d(v, u) = 0 \), which implies \( u = v \). Hence fixed point is unique.

**Remark 3.2** In Theorem 3.1:
(1) If we put \( \eta = 0 \), we obtain Theorem 2.13 of R. Shrivastava et.al. in [7].
(2) If we put \( \beta = \gamma = \eta = 0 \), we obtain Theorem 3.2 of Isufati [1].
(3) If we put \( \beta = \eta = 0 \), we obtain Theorem 3.5 of Aage and Salunke [2].
(4) If we put \( \gamma = \delta = \eta = 0 \), we obtain Theorem 2.12. of [7].

**Theorem 3.3** Let \((X, d)\) be a complete dislocated metric space. Let \( S, T : X \to X \) be continuous mappings satisfying the condition:
for all $x, y \in X$ and $0 < h < \frac{1}{2}$. Then $S$ and $T$ have unique common fixed point.

**Proof.** Let $x_0 \in X$ be arbitrary. Define the sequence $(x_n)_{n \in \mathbb{N}}$ inductively:
\[ x_0 = S(x_0), x_1 = T(x_0), \ldots, x_{2n} = T(x_{2n-1}), x_{2n+1} = S(x_{2n}). \]

By the condition we have:
\[
\begin{align*}
  d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\
  &\leq h \max \left\{ \frac{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1})} \right\} \\
  &= h \max \left\{ \frac{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1})} \right\} \\
  &= h \max \left\{ \frac{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1})} \right\} \\
  &\leq h (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))
\end{align*}
\]

Therefore
\[
\begin{align*}
  d(x_{2n+1}, x_{2n+2}) &\leq h \frac{1}{1-h} d(x_{2n}, x_{2n+1}) \\
  &\text{define } r = \frac{h}{1-h}, \quad 0 < r < 1
\end{align*}
\]
continuing in this way we get
\[ d(x_{2n+1}, x_{2n+2}) \leq r^n d(x_0, x_1), \quad \text{since } 0 < r < 1, \quad r^{2n} \to 0, \quad \text{for } n \to \infty. \]

Hence, $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in complete dislocated metric space $(X, d)$. So there exists $u \in X$, such that $(x_n)_{n \in \mathbb{N}}$ converges to $u$. Further, the subsequences $(Sx_{2n}) \to u$ and $(Tx_{2n+1}) \to u$. Since $S, T : X \to X$ are continuous, will have $Su = u$ and $Tu = u$.

**Uniqueness:** Let $u$ and $v$ be fixed points of $S, T$.

Then:
\[
\begin{align*}
  d(u, v) &= d(Su, Tv) \\
  &\leq h \max \left\{ d(u, v), d(u, Su), d(v, Tv), d(u, Tv), d(v, Su), d(u, Su)d(v, Tv) \right\} \\
  &= h \max \left\{ d(u, v), d(u, u), d(v, v), d(u, v), d(u, v), d(u, v) \right\}
\end{align*}
\]
Replacing \( v \) by \( u \) in (7), we get: 
\[ d(u, u) \leq h d(u, u), \text{ since } 0 < h < \frac{1}{2}. \]
Hence:
\[ d(u, u) = 0 \quad (8) \]
Similarly can show
\[ d(v, v) = 0 \quad (9) \]
Again from (7) \( d(u, v) \leq h d(u, v) \), which implies that \( d(u, v) = 0 \), since \( (X, d) \) is dislocated we have \( u = v \).

**Example 3.4** Let \( X = \{1, 2, 3\} \) and \( d(x, y) = \begin{cases} 2, & \text{if } x + y \text{ is even} \\ 1, & \text{if } x + y \text{ is odd} \end{cases} \)
Define \( S, T : X \to X \) as \( S(1) = S(2) = S(3) = 1 \) and \( T(1) = T(2) = T(3) = 2 \)
we observed
\[ d(Sx, Ty) = \frac{1}{2} \max \left\{ \frac{d(x, y)}{d(x, Sx)}, \frac{d(x, Ty)}{d(x, Ty)} \right\} \]
for all \( x, y \in X \).

We note that Theorem 3.3 is not valid for \( h = \frac{1}{2} \), because \( S \) and \( T \) have no common fixed points. Clearly 1 is a fixed point of \( S \), and 2 is a fixed point of \( T \).

**Remark 3.5** If we put \( S = T \) we obtain result for a continuous mapping \( T \) on \( X \) and the contractive condition of Theorem 3.3 is more general.

**References**


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