Correct Mean Value Interpolation on Parabola II

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Abstract

In this paper bivariate mean-value interpolation problem, where interpolation parameters are integrals over same radius circles which their centers are on a parabola unless one of them not on it is considered. In this case the correctness of the problem is discussed.

Keywords: Bivariate polynomials, Correct interpolation, Mean-value interpolation, parabola

1 Introduction

Denote by $\Pi_n = \Pi_n(\mathbb{R}^2)$ the space of bivariate polynomials of total degree not exceeding $n$:

$$\Pi_n = \{ p(x,y) = \sum_{i+j \leq n} a_{ij} x^i y^j : i, j \in \mathbb{Z}_+ \}.$$ 

Set $N := dim \Pi_n = \binom{n+2}{2}$.

Let us fix the set of distinct points

$$\mathcal{X}_s = \{(x_1, y_1), \ldots, (x_s, y_s)\} \subset \mathbb{R}^2$$

as the set of interpolation nodes.

The classic Lagrange interpolation problem $(\Pi_n, \mathcal{X}_s)$ is to find a unique polynomial $p \in \Pi_n$ such that

$$p(x_k, y_k) = c_k, \quad k = 1, \ldots, s,$$  \hspace{1cm} (1)

where $c_k, k = 1, \ldots, s$ are real numbers.

Definition 1.1. The Lagrange pointwise interpolation problem $(\Pi_n, \mathcal{X}_s)$ is called correct(poised) if, for any real values $c_k, k = 1, \ldots, s$, there exists a unique polynomial $p \in \Pi_n$ satisfying the conditions (1).
In other words, the Lagrange interpolation problem is to find a unique polynomial \( p(x, y) = \sum_{i+j \leq n} a_{ij} x^i y^j \in \Pi_n \) which reduce the conditions (1) to the following linear system

\[
p(x_k, y_k) = \sum_{i+j \leq n} a_{ij} x_k^i y_k^j = c_k, \quad k = 1, \ldots, s.
\] (2)

The correctness of interpolation means that the linear system (2) has a unique solution for arbitrary right hand side values. A necessary condition for this is \( s = N \). We know that in this case the linear system (2) has a unique solution for arbitrary values \( \{c_1, \ldots, c_s\} \) if and only if the corresponding homogeneous system has only trivial solution.

**Theorem 1.2** The interpolation problem \((\Pi_n, X_N)\) is correct if and only if

\[
p \in \Pi_n, \quad p(x_k, y_k) = 0, \quad k = 1, \ldots, N \Rightarrow p = 0.
\]

From now on, let

\[
D := \{D_k : k = 1, \ldots, N\}
\]

be a collection of Lebesgue measurable sets of finite non-zero measure. In this paper a mean-value interpolation problem is considered where interpolation parameters are integrals over circles which their centers are on a parabola. Here we are going to find a unique polynomial \( p \in \Pi_2 \) such that

\[
\frac{1}{\mu_2(D_l)} \int \int_{D_l} p(x, y) dxdy = c_l, \quad l = 1, \ldots, 6,
\] (3)

where \( c_k \)'s are arbitrary given numbers and \( D_l \)'s are circles and also \( \mu_2(D_l) \) is is the area of \( D_l \). Denote this interpolation problem by \((\Pi_2, D)^{m.v.}\), where \( D \) is the set of above circles: \( D = \{D_i : i = 1, \ldots, 6\} \).

Same as Definition 1.1 we call the problem \((\Pi_2, D)^{m.v.}\) correct if, for any numbers \( c_l \)'s, \( l = 1, \ldots, 6 \) there exists a unique polynomial \( p \in \Pi_2 \) satisfying (3). In the sequel we will use the following well known

**Theorem 1.3** The mean-value interpolation problem \((\Pi_n, D)^{m.v.}\) is correct if and only if

\[
p \in \Pi_n, \int \int_{D_k} p(x, y) dxdy = 0, \quad k = 1, \ldots, N \Rightarrow p = 0.
\]

An example of correct interpolation problem in dimension two is presented in [2, 3]. This case is considered for degree not exceeding one.

**Lemma 1.4** The problem in \( \mathbb{R}^2 \) with four arbitrary regions and \( n=1 \) is correct if and only if the centroids of the regions are not laying on a hyperplane.
Another special case is considered in [5]. In [5] we consider an arbitrary set of $N$ distinct balls of same radius $r$:

$$D = B := \{ B_r(a_i) : i = 1, ..., N \},$$

where $B_r(a_i)$ is the circle of radius $r \in \mathbb{R}^+$ centered at $a_i \in \mathbb{R}^2$. Let $A = \{ a_i, i = 1, \ldots, N \}$ be the set of centers of the balls. We have

**Theorem 1.5** The mean-value interpolation $(\Pi_n, B)^{m.v.}$ is poised if and only if the Lagrange pointwise interpolation problem $(\Pi_n, A)$ is poised.

It is worth mentioning that in [5] Theorem 1.5 is proved in arbitrary dimension. For other versions of mean-value interpolation problem we refer to [1-5].

## 2 Preliminary Notes

Let us consider the classic interpolation with polynomials of degree two and the nodes are on the above parabola. Namely, six interpolation points in $\mathbb{R}^2$ on parabola such that this parabola has equation $P : y = x^2$.

Of course the interpolation problem corresponds to the above points is not poised, because it is enough to get the interpolation polynomial by the following

$$p(x, y) = y - x^2.$$ 

Then in view of (2) the corresponding homogeneous system has non-trivial solution.

Now let us consider the following sets which are circles that their centers on a parabola. It is easily seen that if parabola is the form $y = mx^2 + n$, where $m, n$ are numbers in $\mathbb{R}$ then one can shift to $P$. Therefore, it is enough to check the parabola $P : y = x^2$. Thus, consider the mean-value interpolation problem $(\Pi_2, D)^{m.v.}$ over the above circles.

## 3 Main Results

The following theorem is the result of this paper.

**Theorem 3.1** Suppose that six circles on parabola $P : y = x^2$ are given. Let any two centers of circles be on $P$. Suppose also two of them on one line parallel to $x$-axis are different radii and the remaining circles are same radii. Then the mean-value interpolation problem $(\Pi_2, D)^{m.v.}$ is correct.

**Proof.** To prove the mean-value interpolation is correct, according to proposition 1.1 it is enough to show

$$\forall p, p \in \Pi_2, \int \int_{D_i} p(x, y) dxdy = 0, \quad l = 1, \ldots, 6 \Rightarrow p = 0,$$  \quad (4)
where $D_l$'s are the above circles.

One can compute the integral over circles with same radii

$$
\int \int_{D_l} p(x, y) dxdy = \int \int_{D_l} \sum_{i+j \leq 2} a_{ij} x^i y^j dxdy =
$$

$$
= r^2 \sum_{i+j \leq 2} a_{ij} \int \int_{D'_{x^2+y^2 \leq 1}} (rx + \alpha_1)^i (ry + \beta_1)^j dxdy,
$$

where $D_l, l = 1, \ldots, 4$ are are circles with same radii on $P$. Hence

$$
\int \int_{D_l} p(x, y) dxdy = \pi r^2 [a_{00} + \alpha_1 a_{10} + \beta_1 a_{01} + a_{20}(\frac{1}{4} r^2 + \alpha_1^2) +
$$

$$+ a_{11} \alpha_1 \beta_1 + a_{02}(\frac{1}{4} r^2 + \beta_1^2)], l = 1, \ldots, 4.
$$

Now for different radii circles we have

$$
\int \int_{D_l} p(x, y) dxdy = \int \int_{D_l} \sum_{i+j \leq 2} a_{ij} x^i y^j dxdy =
$$

$$
= r^2 \sum_{i+j \leq 2} a_{ij} \int \int_{D'_{x^2+y^2 \leq 1}} (r_i x + \alpha_i)^i (r_i y + \beta_i)^j dxdy, l = 1, 2.
$$

In view of the hypothesis the integral is as follows

$$
\int \int_{D_l} p(x, y) dxdy = \pi r_i^2 [a_{00} + \alpha_i a_{10} + \beta_i a_{01} + a_{20}(\frac{1}{4} r_i^2 + \alpha_i^2) +
$$

$$+ a_{11} \alpha_i \beta_i + a_{02}(\frac{1}{4} r_i^2 + \beta_i^2)], l = 1, 2.
$$

Assume that the centers of circles are the following points

$$(\alpha, \alpha^2), (-\alpha, \alpha^2), (\beta, \alpha^2), (-\beta, \alpha^2), (\gamma, \gamma^2), (-\gamma, \gamma^2).$$

Using these points and the fact that the corresponding homogenous linear system (4) is correct if and only if the corresponding Vandermonde determinant of right hand side values is not equal to zero we have

$$
Det A = \pi^6 r_1^2 r_2^2 r_3^2 Det
$$

$$
\begin{pmatrix}
1 & \alpha & \alpha^2 & \frac{1}{4} r_1^2 + \alpha^2 & \alpha^3 & \frac{1}{4} r_1^2 + \alpha^4 \\
1 & -\alpha & -\alpha^2 & \frac{1}{4} r_2^2 + \alpha^2 & -\alpha^3 & \frac{1}{4} r_2^2 + \alpha^4 \\
1 & \beta & \beta^2 & \frac{1}{4} r_3^2 + \beta^2 & \beta^3 & \frac{1}{4} r_3^2 + \beta^4 \\
1 & -\beta & -\beta^2 & \frac{1}{4} r_3^2 + \beta^2 & -\beta^3 & \frac{1}{4} r_3^2 + \beta^4 \\
1 & \gamma & \gamma^2 & \frac{1}{4} r_3^2 + \gamma^2 & \gamma^3 & \frac{1}{4} r_3^2 + \gamma^4 \\
1 & -\gamma & -\gamma^2 & \frac{1}{4} r_3^2 + \gamma^2 & -\gamma^3 & \frac{1}{4} r_3^2 + \gamma^4
\end{pmatrix}
$$
After some simplification and by expanding the determinant we finally have

\[ \text{Det}A = \pi^6 r_1^2 r_2^2 r_3^8 \beta (\beta^2 - \alpha^2) \gamma (\alpha^2 - \gamma^2) (\beta^2 - \gamma^2)^2 (r_1^2 - r_2^2). \]

The determinant is not equal to zero, because by assumption each \( \alpha \neq 0, \beta \neq 0, \gamma \neq 0, \alpha \neq -\beta, \beta \neq \gamma, \beta \neq -\gamma, \alpha \neq \gamma, \alpha \neq -\gamma, r_1 \neq r_2. \)

Therefore \( \text{Det}A \neq 0 \) Hence, the mean-value interpolation problem \( (\Pi_2, D)^{m.v.} \) is correct.

References


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