Solving Linear Two-Point Boundary Value Problems by Direct Adams Moulton Method

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Abstract

In this paper, we present direct method of Adams Moulton type for solving second order linear two-point boundary value problems (LBVPs) directly. LBVPs will be replaced by two initial value problems (IVPs) and those IVPs will be solved using direct method. Then the linear shooting technique is used to the resulting equation to construct the numerical solution. The method will be implemented using constant step size. Numerical results are given to illustrate the performance of the direct method compared to the existing methods.

Mathematics Subject Classification: 65L06, 65L10

Keywords: Linear boundary value problem, Direct method, Linear shooting method

1 Introduction

The boundary value problems of ordinary differential equation play a significant role in wide variety of problems such as electrostatic potential between two concentric metal, chemical reaction, heat transfer and deflection of a beam. These problems can be presented by using boundary value problem with two boundary conditions.

This paper is concerned for solving LBVPs that involve second order differential equation given as follows:
The Adomian decomposition method has been widely cited by many researchers for solving differential problems. However, it has difficulties in dealing with boundary conditions for solving two-point boundary value problems. However, Bongsoo [2] has introduced a new Adomian decomposition method known as extended Adomian decomposition method in solving linear and non-linear two-point boundary value problems. This EADM has improved the ADM results by creating a new canonical form containing both boundary conditions \( a \) and \( b \). Jafri et al. [3] has considered solving directly two-point boundary value problems for second order ordinary differential equations using multistep methods in the form of backward difference formulas. Faires and Burden [1] also using the same idea where the authors have reformulated equation (1) into the system of first order ordinary differential equations and then apply fourth order Runge-Kutta method together with linear shooting method to solve LBVPs. The advantage of this reducing method is greatly improving the accuracy of approximation solution compare to the exact solution. While, Taiwo [5] has implemented a simple algebraic technique using collocation points to solve LBVPs. As the collocation points increase, the errors of the approximation become smaller. Hamid et al. [4] had used extended cubic B-spline to solve LBVPs by transforming the LBVPs into piecewise polynomial function. The author has improved the results by using only five or six iteration to ensure the method will converge. Majid [8] has solved the nonlinear two point boundary value problem using direct method of Adams Moulton type.

In this paper, we are going to solve two-point LBVPs directly by combining the linear shooting method with the direct method of Adams Moulton type in Majid [8].

**Theorem:**
Suppose the function \( f(x, y, y') = P(x)y' + Q(x)y + R(x) \) in (1) is continuous on the set
\[
D = \{(x, y, y') | a \leq x \leq b, -\infty < y < \infty, -\infty < y' < \infty\}
\]
and the partial derivatives \( f_y \) and \( f_{y'} \) are also continuous on \( D \). If
(i) \( f_y(x, y, y') > 0 \), for all \( (x, y, y') \in D \), and
(ii) a constant \( M \) exist, with
\[
|f_{y'}(x, y, y')| \leq M, \text{ for all } (x, y, y') \in D
\]
then the boundary value problem has a unique solution.

**Corollary:**
If the LBVPs in (1) satisfies
Solving linear two-point boundary value problems

(i) \( p(x),\ q(x),\) and \( r(x) \) are continuous on \([a, b],\)

(ii) \( q(x) > 0 \) on \([a, b],\)
then the problem has a unique solution.

All the LBVPs tested must first ensure that the solutions exist and have a unique solution.

2 Formulation and implementation

The formulae of \( y_{n+1}, \) the direct method of Adams Moulton method at the point \( x_{n+1} \) can be obtained by integrating (1) once and twice as follows

\[
\text{Integrate once:} \quad \int_{x_n}^{x_{n+1}} y'(x) \, dx = \int_{x_n}^{x_{n+1}} f(x, y, y') \, dx. \tag{2}
\]

Therefore,

\[
y'(x_{n+1}) = y'(x_n) + \int_{x_n}^{x_{n+1}} f(x, y, y') \, dx. \tag{3}
\]

\[
\text{Integrate twice:} \quad \int_{x_n}^{x_{n+1}} \int_{x_n}^{x} y''(x) \, dx \, dx = \int_{x_n}^{x_{n+1}} \int_{x_n}^{x} f(x, y, y') \, dx \, dx. \tag{4}
\]

Therefore,

\[
y(x_{n+1}) - y(x_n) - hy'(x_n) = \int_{x_n}^{x_{n+1}} f(x, y, y') \, dx. \tag{5}
\]

The function \( f(x, y, y') \) in (3) and (5) will be approximated using Lagrange interpolation polynomial of order four and five where the interpolation points involved are four and five points i.e. \( \{x_{n-3}, x_{n-2}, x_{n-1}, x_n, x_{n+1}\} \) and \( \{x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}, x_n, x_{n+1}\} \) respectively. Taking \( s = \frac{x - x_{n+1}}{h} \) and by replacing \( dx = h \, ds \), the value of \( x_{n+1} \) can be easily obtained by integrating (3) and (5) over the interval \([x_n, x_{n+1}]\) using Maple software. This method is combination of predictor and corrector where the predictor formula is one order less than the corrector. Evaluate these integrals using MAPLE and we will obtain the corrector formulae as follows:

**Direct method of order 4 (1PDAM4)**

**Corrector formula**

\[
y_{n+1}' = y_n' + \frac{h}{24}(29f_{n+1} + 20f_n - 5f_{n-1} + f_{n-2})
\]
\[ y_{n+1} = y_n + hy'_n + \frac{h^2}{360} (38f_{n+1} + 171f_n - 36f_{n-1} + 7f_{n-2}) \]

**Direct method of order 5 (1PDAM5)**

**Corrector formula**

\[ y'(x_{n+1}) = y'(x_n) + \frac{1}{720} h(251f_{n+1} + 646f_n - 264f_{n-1} + 106f_{n-2} - 19f_{n-3}) \]
\[ y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{1}{1440} h^2 (135f_{n+1} + 752f_n - 246f_{n-1} + 96f_{n-2} - 17f_{n-3}) \]

For calculation of the initial points, two methods are involved which is Euler and Modified Euler method. These methods will be used at the beginning of the code to find the starting initial values. Both methods will solve the problems directly. The predictor and corrector direct method then can be implemented until the end of boundary interval. In order to get better approximation for initial values, the value for \( h \) will be reduce to \( \frac{h}{4} \) for 1PDAM4 and \( \frac{h}{8} \) for 1PDAM5. While predictor and corrector direct method will remain using the chosen step size \( h \).

Linear shooting method is based on replacement of the LBVPs (1) by two (IVPs) shown as follows:

\[ y_1'' = P(x)y_1' + Q(x)y_1 + R(x), \quad y_1(a) = \alpha, \quad y_1'(a) = 0 \]
\[ y_2'' = P(x)y_2' + Q(x)y_2, \quad y_2(a) = 0, \quad y_2'(a) = 1 \quad (6) \]

a new function will be constructed by solving the two IVPs:

\[ y = y_1 + \theta y_2 \quad (7) \]

with

\[ y(b) = y_1(b) + \theta y_2(b) = \beta \quad (8) \]

Hence,

\[ \theta = \frac{\beta - y_1(b)}{y_2(b)} \quad (9) \]

using the linear shooting method (7), a series of approximation solutions to the LBVPs will be calculated from the new function.

### 3 Numerical results

In this section, we demonstrate the effectiveness of the direct method with three tested problems.
**Problem 1:**
\[
\frac{d^2 y}{dx^2} = y + \cos(x), \quad 0 \leq x \leq 1
\]
Boundary condition: \( y(0) = 1, y(1) = 1 \)
Exact solution:
\[
y = \frac{-3 \cosh(1) + 3 \sinh(1) + \cos(1) + 2}{4 \sinh(1)} e^x + \frac{3 \cosh(1) + 3 \sinh(1) - \cos(1) - 2}{4 \sinh(1)} e^{-x} - \cos(x)
\]
Source: Bongsoo [2]

**Problem 2:**
\[
\frac{d^2 y}{dx^2} = \frac{dy}{dx} - \exp(x - 1) - 1, \quad 0 \leq x \leq 1
\]
Boundary condition: \( y(0) = 0, y(1) = 0 \)
Exact solution: \( y(x) = x(1 - \exp(x - 1)) \)
Source: Wang et al. [6]

**Problem 3:**
\[
\frac{d^2 y}{dx^2} = -(x + 1) \frac{dy}{dx} + 2y + (1 - x^2)e^{-x}, \quad 0 \leq x \leq 1
\]
Boundary condition: \( y(0) = -1, y(1) = 0 \)
Exact solution: \( y(x) = (x - 1)e^{-x} \)
Source: Hamid et al. [4]

The following notations are used in the tables:

- **EADM**: Extended Adomian decomposition method in Bongsoo [2]
- **ECBIM**: Extended cubic B-spline method in Hamid et al. [4]
- **HPM**: Homotopy perturbation method of order four in Wang et al. [6]
- **RK4**: Runge-Kutta method of order four in Faires and Burden [1]
- **1PDAM4**: One point direct method of order 4
- **1PDAM5**: One point direct method of order 5
- *****: Maximum error

The results were displayed in Table 1 – 6. In Table 1 we could observed that 1PDAM4 and 1PDAM5 are able to obtain comparable accuracy compared to EADM. EADM required five iterations to obtain the given result while 1PDAM4 and 1PDAM5 will solve without any iteration. Hence, 1PDAM4 and 1PDAM5 can solve faster compare to EADM.
Table 1: Local errors of EADM, 1PDAM4 and 1PDAM5 for solving problem 1 when $h = 0.125$

<table>
<thead>
<tr>
<th>$x$</th>
<th>EADM in [2]</th>
<th>1PDAM4</th>
<th>1PDAM5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>4.37e-7</td>
<td>1.69e-6</td>
<td>1.46e-6</td>
</tr>
<tr>
<td>2/8</td>
<td>8.07e-7</td>
<td>2.25e-6*</td>
<td>1.79e-6*</td>
</tr>
<tr>
<td>3/8</td>
<td>1.05e-6</td>
<td>2.23e-6</td>
<td>1.53e-6</td>
</tr>
<tr>
<td>4/8</td>
<td>1.14e-6*</td>
<td>2.07e-6</td>
<td>1.20e-6</td>
</tr>
<tr>
<td>5/8</td>
<td>1.05e-6</td>
<td>1.78e-6</td>
<td>8.84e-7</td>
</tr>
<tr>
<td>6/8</td>
<td>8.07e-7</td>
<td>1.34e-6</td>
<td>5.84e-7</td>
</tr>
<tr>
<td>7/8</td>
<td>4.37e-7</td>
<td>7.53e-7</td>
<td>2.91e-7</td>
</tr>
</tbody>
</table>

Table 2: Local errors of HPM, 1PDAM4 and 1PDAM5 for solving problem 2 when $h = 0.1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>HPM in [7]</th>
<th>1PDAM4</th>
<th>1PDAM5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>5.17e-06</td>
<td>2.20e-6</td>
<td>1.69e-6</td>
</tr>
<tr>
<td>0.2</td>
<td>8.97e-06</td>
<td>3.52e-6</td>
<td>2.43e-6</td>
</tr>
<tr>
<td>0.3</td>
<td>1.00e-05*</td>
<td>4.18e-6</td>
<td>2.47e-6*</td>
</tr>
<tr>
<td>0.4</td>
<td>7.94e-06</td>
<td>4.65e-6</td>
<td>2.30e-6</td>
</tr>
<tr>
<td>0.5</td>
<td>3.59e-06</td>
<td>4.89e-6*</td>
<td>2.09e-6</td>
</tr>
<tr>
<td>0.6</td>
<td>1.44e-06</td>
<td>4.83e-6</td>
<td>1.83e-6</td>
</tr>
<tr>
<td>0.7</td>
<td>5.29e-06</td>
<td>4.40e-6</td>
<td>1.50e-6</td>
</tr>
<tr>
<td>0.8</td>
<td>6.47e-06</td>
<td>3.53e-6</td>
<td>1.09e-6</td>
</tr>
<tr>
<td>0.9</td>
<td>4.46e-06</td>
<td>2.10e-6</td>
<td>6.00e-7</td>
</tr>
</tbody>
</table>

Table 2 shows the comparison of local errors at each step between HPM and the direct methods. The direct methods gave comparable results but yet smaller value of maximum error compared to HPM.

Table 3: Results of EADM, 1PDAM4 and 1PDAM5 for solving problem 2 when $h = 0.01$

<table>
<thead>
<tr>
<th>Method</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>EADM in [2]</td>
<td>1.05e-10</td>
</tr>
<tr>
<td>1PDAM4</td>
<td>5.72e-10</td>
</tr>
<tr>
<td>1PDAM5</td>
<td>2.06e-10</td>
</tr>
</tbody>
</table>

In Table 3, we could observed that the direct methods give comparable results compared to EADM when the step size $h = 0.01$ for solving problem 2. Table 4 shows the local errors at each steps for 1PDAM4 and 1PDAM5 when $h = 0.1$. 
Table 4: Local errors of 1PDAM4 and 1PDAM5 for solving problem 3 when \( h = 0.1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1PDAM4</th>
<th>1PDAM5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.19e-6</td>
<td>3.67e-6</td>
</tr>
<tr>
<td>0.2</td>
<td>6.27e-6</td>
<td>5.10e-6*</td>
</tr>
<tr>
<td>0.3</td>
<td>6.89e-6</td>
<td>4.93e-6</td>
</tr>
<tr>
<td>0.4</td>
<td>7.25e-6</td>
<td>4.42e-6</td>
</tr>
<tr>
<td>0.5</td>
<td>7.31e-6*</td>
<td>3.90e-6</td>
</tr>
<tr>
<td>0.6</td>
<td>6.98e-6</td>
<td>3.31e-6</td>
</tr>
<tr>
<td>0.7</td>
<td>6.19e-6</td>
<td>2.65e-6</td>
</tr>
<tr>
<td>0.8</td>
<td>4.83e-6</td>
<td>1.89e-6</td>
</tr>
<tr>
<td>0.9</td>
<td>2.81e-6</td>
<td>1.02e-6</td>
</tr>
</tbody>
</table>

Table 5: Results of ECBIM, 1PDAM4 and 1PDAM5 for solving problem 3 when \( h = 0.1 \)

<table>
<thead>
<tr>
<th>Method</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>ECBIM in [4]</td>
<td>6.61e-6</td>
</tr>
<tr>
<td>1PDAM4</td>
<td>7.31e-6</td>
</tr>
<tr>
<td>1PDAM5</td>
<td>5.10e-6</td>
</tr>
</tbody>
</table>

Table 5 shows that the maximum error of the direct method is comparable to the ECBIM when \( h = 0.1 \). ECBIM require five to six iterations for the method to converge while the direct method solved the LBVPs without any iteration. Hence, the direct method can solve faster compare to ECBIM.

Table 6: Results of RK4, 1PDAM4 and 1PDAM5 for solving problem 1, 2 and 3

<table>
<thead>
<tr>
<th>Step size</th>
<th>Method</th>
<th>Function Call</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Problem 1</td>
</tr>
<tr>
<td>0.1</td>
<td>1PDAM4</td>
<td>66</td>
<td>9.701185e-007</td>
</tr>
<tr>
<td></td>
<td>1PDAM5</td>
<td>66</td>
<td>7.750486e-007</td>
</tr>
<tr>
<td></td>
<td>RK4</td>
<td>88</td>
<td>2.738664e-007</td>
</tr>
<tr>
<td>0.05</td>
<td>1PDAM4</td>
<td>106</td>
<td>6.550903e-008</td>
</tr>
<tr>
<td></td>
<td>1PDAM5</td>
<td>106</td>
<td>5.513988e-008</td>
</tr>
<tr>
<td></td>
<td>RK4</td>
<td>168</td>
<td>1.688936e-008</td>
</tr>
<tr>
<td>0.01</td>
<td>1PDAM4</td>
<td>426</td>
<td>1.094134e-010</td>
</tr>
<tr>
<td></td>
<td>1PDAM5</td>
<td>426</td>
<td>1.004481e-010</td>
</tr>
<tr>
<td></td>
<td>RK4</td>
<td>808</td>
<td>2.676692e-011</td>
</tr>
</tbody>
</table>
Table 6 display the advantage of 1PDAM4 and 1PDAM5 compared to RK4. The total function call is lesser for the direct methods at all tested step sizes. However the RK4 will reduce the tested problems to first order ODEs. Hence, we could conclude that the direct methods are less expensive compared to RK4. In term of accuracy, the direct methods can generate comparable accuracy even with less function call.

4 Conclusion

In this paper, we have shown the proposed direct method of Adams Moulton type with linear shooting technique using constant step size is suitable for solving second order linear two-point boundary value problems (LBVPs).

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References


differential equations using two point four step direct implicit block method.

[8] Z.A. Majid, P.P. See and M. Suleiman. Solving Directly Two point Non
Linear Boundary value problem Using Direct Adams Moulton Method,

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