Determination of Optimal Reserve between Two
Machines in Series with Truncation Point in Repair
Time Distribution

S. Srinivasan\textsuperscript{1}, P. Sheik Uduman\textsuperscript{2} and R. Sathiyaamoorthy\textsuperscript{3}

\textsuperscript{1}Department of Mathematics, B.S. Abdur Rahman University
\textsuperscript{2}Department of Mathematics, B.S. Abdur Rahman University,
Chennai – 600 048, Tamil Nadu, India
\textsuperscript{3}Department of Statistics(Retd), Annamalai University

Abstract

Proper planning of inventory size is an attempt to increase probability in any production industry or marketing organization. When the arrangement and functioning of machines in a factory forms a series system, then the determination of the optimal size of semi finished products between machines in series becomes imperative. In this paper the determination of the optimal reserve inventory between two machines $M_1$ and $M_2$ in series is discussed. It is assumed that the repair time of $M_1$ is considered as a random variable and its distribution undergoes a change after a particular value of the random variable which otherwise is called the truncation point. Numerical illustrations are also given.

Keywords: Series system, Semi finished products, Change point

INTRODUCTION

Ramanarayanan[2] considered a Markovian inventory system and had obtained the explicit steady state solution. In his model the maximum inventory capacity is $S$ and reorder level is $s$. The demands occur according to a Poisson process with parameter $\lambda$ and the demand is at the rate of one unit at a time. The lead-time distribution is according to the phase type (PH).

Chenniappan etal [6] considered a new type of an inventory situation. In his model the inventories are kept as two different stocks. When a demand occurs, one unit from each of the two inventories is sold. The model is such that the order for the first product is supplied along with the second product. Sometimes the first
product alone is supplied without the second product. For example, computers are sold with or without a printer. Considering the distributions of the inter-arrival times between demands as following (i) Exponential distribution and (ii) General distributions, the authors have derived the steady state probabilities for the inventory levels. It may be noted that the scope of this problem here is only to find out probability of different inventory levels; optimal solution has not been attempted.

Hadly and Whitin[8] considered a similar inventory model in which the concept of salvage loss due to unsold items within the prescribed time limit is considered. According to this property the pdf of the random variable denoting the repair time of machine $M_1$ undergoes a parametric change at a truncation point denoted as '$x_0$' which is a constant; and so the pdf is $g(t, \theta)$ if $t \leq x_0$ and it is $g(t, \theta')$ if $t > x_0$. The optimal size of the reserve inventory under these assumptions has been derived. The authors have derived the expression for optimal reserve inventory under the assumption that truncation point '$x_0$' is a random variable, which follows an exponential distribution. This result is discussed by Sachithanantham et al[3].

Sehik Uduman et al[9] considered the problem of the determination of the optimal reserve inventory between two machines in series using the concept of order statistics. In this model, they have taken up the problem in which two machines are in series so that the output of machine $M_1$ happens to be the input for machine $M_2$. Whenever the breakdown of $M_1$ occurs the supply of raw material to $M_2$ is stopped. Then $M_2$ becomes idle which proves costly. The expression for optimal reserve inventory between the two machines has been derived, taking into consideration the inventory holding cost and also the idle time cost of $M_2$. In doing so the authors have introduced the distributions of the order statistics to find the mean interarrival time between successive breakdowns of machine $M_1$. The expression for optimal inventory has been obtained both in the case of the first order statistic as well as the $n^{th}$ order statistic which represent the interarrival times between the breakdowns of machine $M_1$.

In this paper a model with two machines in series is considered. The output of machine $M_1$ is the input for machine $M_2$. Whenever there is a breakdown of machine $M_1$, the input for machine $M_2$ is not possible and so the machine $M_2$ is forced to be in idle state. The idle time of $M_2$ is very costly. Hence the reserve inventory of the semi finished products in between $M_1$ and $M_2$ is suggested. If the inventory is in excess of requirement it involves holding cost. On the other hand if there is shortage of inventory, there is an idle time cost for machine $M_2$ which is known as the shortage cost.
The consumption rate of $M_2$ is taken as a constant $r$. The repair time or the down time of $M_1$ is a random variable denoted as $\tau$, with $g(\tau)$ as the pdf.

It may be noted that if $S$ is the size of the reserve inventory then the idle time of $M_2$ is given by

$$t = \begin{cases} 
0 & \text{if } \tau \leq \frac{S}{r} \\
\tau - \frac{S}{r} & \text{if } \tau > \frac{S}{r}
\end{cases} 
.......... (1.1)$$

The expected cost of overages and shortages in inventory is given by the equation

$$E(C) = hS + \frac{d}{\mu} \int_{S/r}^{\infty} \left[ e - \frac{x}{r} \right] g(x) \text{d}x 
.......... (1.2)$$

where $\frac{1}{\mu} = \text{the average number of breakdowns of } M_1 \text{ per unit time.}$

$h = \text{inventory holding cost.}$

$d = \text{shortage cost.}$

The optimal reserve size $\hat{S}$ can be obtained by solving the equation

$$\frac{dE(C)}{dS} = 0$$

Hence

$$G \left[ \frac{\hat{S}}{r} \right] = 1 - \left[ \frac{r \mu h}{d} \right] 
.......... (1.3)$$

This is the basic model discussed in Hanssman [1].

A modification of this model by introducing the so-called the Setting the Clock Back to Zero (SCBZ) property has been attempted by Sachithanandam et al [3]. A different version of this model has been discussed by Sathiyamoorthi et al [2].

**THE MODEL**

**ASSUMPTIONS**

i) There are two machines in series and the out put of machine $M_1$ is the input for the $M_2$.

ii) The consumption rate of $M_2$ is a constant denoted by $r$.

iii) The repair time of $M_1$ is a random variable and it undergoes a change of distribution after a truncation point over the time axis. The repair time of $M_1$ follows exponential distribution with parameter $\theta_1$ before the truncation point and Erlang 2 distribution after the truncation point.

**NOTATIONS**

$\tau = \text{a random variable denoting the repair time of } M_1$

$\tau \sim \text{exponential with parameter } \theta_1 \text{ if } \tau \leq t$
\[ \tau \sim \text{Erlang 2 with parameter } \theta_2 \text{ if } \tau > t. \]
\[ t = \text{truncation point or change point.} \]
\[ g(\tau) = \text{the pdf of } \tau. \]
\[ h = \text{the inventory holding cost.} \]
\[ d = \text{the idle time cost of } M_2. \]
\[ \frac{1}{\mu} = \text{average number of breakdowns per unit of time of } M_1. \]
\[ S = \text{size of the reserve inventory between } M_1 \text{ and } M_2. \]
\[ \hat{S} = \text{the optimal size of reserve inventory between } M_1 \text{ and } M_2. \]

**RESULTS**

In this model it is assumed that the random variable \( \tau \) with pdf \( g(\tau) \) undergoes change of distribution after a change point \( 't' \). If \( \tau \leq t \) then \( \tau \sim \exp(\theta_1) \) and if \( \tau > t \) then \( \tau \sim \exp(\theta_2) \) with parameter \( \theta_2 \). The truncation point \( 't' \) itself is a random variable which follows \( \exp(\lambda) \).

Now let \( g_1(\tau) = \theta_1 e^{-\theta_1 \tau} \text{ if } 0 < \tau < t \)
\[ g_1(\tau) = \theta_1 e^{-\theta_1 \tau} P[\tau < t] \]
\[ g_1(\tau) = \theta_1 e^{-\tau(\theta_1 + \lambda)} \]

and
\[ g_2(\tau) = \theta_2^2 (\tau - t) e^{-\theta_2 (\tau - t)} e^{-\theta_1 \tau} P[\tau > t] \]
\[ \therefore g_2(\tau) = \lambda \theta_2^2 \left[ \frac{e^{-\tau(\theta_1 + \lambda)}}{(\theta_1 - \theta_2 + \lambda)^2} - \frac{e^{-\theta_1 \tau}}{(\theta_1 - \theta_2 + \lambda)^2} + \frac{\tau e^{-\theta_2 \tau}}{(\theta_1 - \theta_2 + \lambda)} \right] \]

**CASE I**

Let us assume that the truncation point \( t \) is such that \( t \leq \frac{S}{r} \). In such a case expression for \( E[C] \) is given by

\[ E[C] = hr \int_0^{S/r} g_1(\tau) d\tau + hr \int_\frac{S}{r}^\infty g_2(\tau) d\tau + \frac{d}{\mu} \int_\frac{S}{r}^\infty \tau - \frac{S}{r} g_2(\tau) d\tau \]

\[ E[C] = hrI_1 + hrI_2 + \frac{d}{\mu} I_3 \]

(say)

... (1.7)

Now

\[ I_1 = \int_0^{S/r} \frac{S}{r} - \tau \theta_1 e^{-\tau(\theta_1 + \lambda)} d\tau \]

by using (1.1) for \( g_1(\tau) \)

\[ = \theta_1 \left[ \frac{S}{r} - \tau \right] e^{-\tau(\theta_1 + \lambda)} \left[ \frac{1}{-r(\theta_1 + \lambda)} + \frac{1}{(\theta_1 + \lambda)^2} \right] - \left[ \frac{S}{r} \right] e^{-\tau(\theta_1 + \lambda)} \left[ \frac{1}{-r(\theta_1 + \lambda)} + \frac{1}{(\theta_1 + \lambda)^2} \right] \]

... (1.7)
Determination of optimal reserve

\[
\frac{dI_1}{dS} = \frac{\theta_1}{r(\theta_1 + \lambda)} \left[ 1 - e^{-r(\theta_1 + \lambda)} \right]
\] ... (1.8)

Now

\[
I_2 = \int \left[ \frac{S}{r} - \tau \right] g_2(\tau) d\tau
\]

\[
= \lambda \theta_2^2 \left[ \int \frac{S - \tau}{(\theta_1 - \theta_2 + \lambda)^2} e^{-r(\theta_1 + \lambda)} d\tau - \int \frac{S - \tau}{(\theta_1 - \theta_2 + \lambda)^2} e^{-r\theta_2} d\tau + \int \frac{S - \tau}{(\theta_1 - \theta_2 + \lambda)^2} e^{-r\theta_1} d\tau \right]
\]

By using (1.2) for \( g_2(\tau) \)

\[
I_2 = \lambda \theta_2^2 \left[ A_1 - B_1 + C_1 \right] \text{ say}
\]

From this we have

\[
\frac{dI_2}{dS} = \lambda \theta_2^2 \left[ \frac{dA_1}{dS} - \frac{dB_1}{dS} + \frac{dC_1}{dS} \right]
\] ... (1.9)

We find out

\[
\frac{dI_2}{dS} = \lambda \theta_2^2 \left[ \frac{1}{r(\theta_1 + \lambda)(\theta_1 - \theta_2 + \lambda)^2} \left\{ e^{-r(\theta_1 + \lambda)} - e^{-\frac{\lambda}{r}(\theta_1 + \lambda)} \right\} + \right.
\]

\[
\int \frac{1}{r\theta_2(\theta_1 - \theta_2 + \lambda)^2} \left\{ e^{-r\theta_2} - e^{-\theta_2} \right\} d\tau - \int \frac{e^{-r\theta_2}}{\theta_2} d\tau + \int \frac{e^{-r\theta_1}}{\theta_2} d\tau - \frac{Se^{-r\theta_2}}{\theta_2} + \frac{e^{-r\theta_1}}{\theta_2} \] ... (1.10)

Again

\[
I_3 = \int \left[ \frac{\tau - S}{r} \right] \lambda \theta_2^2 \left[ \frac{e^{-r(\theta_1 + \lambda)}}{(\theta_1 - \theta_2 + \lambda)^2} - \frac{e^{-r\theta_2}}{(\theta_1 - \theta_2 + \lambda)^2} - \frac{\tau e^{-r\theta_2}}{(\theta_1 - \theta_2 + \lambda)^2} \right] d\tau
\]

\[
= \lambda \theta_2^2 \left[ \int \frac{\tau - S}{r} e^{-r(\theta_1 + \lambda)} d\tau - \int \frac{(\tau - S)}{r(\theta_1 - \theta_2 + \lambda)^2} e^{-r\theta_2} d\tau + \int \frac{(\tau - S)}{r(\theta_1 - \theta_2 + \lambda)^2} e^{-r\theta_1} d\tau \right]
\]

\[
\frac{dI_3}{dS} = \lambda \theta_2^2 \left[ -\frac{e^{-r(\theta_1 + \lambda)}}{r(\theta_1 + \lambda)(\theta_1 - \theta_2 + \lambda)} + \frac{e^{-r\theta_1}}{r\theta_2(\theta_1 - \theta_2 + \lambda)^2} - \frac{1}{r(\theta_1 - \theta_2 + \lambda)} \left[ \frac{Se^{-r\theta_2}}{\theta_2} + \frac{e^{-r\theta_1}}{\theta_2} \right] \right]
\] .... (1.11)
Now
\[
\frac{dE[C]}{dS} = \frac{h\theta_1}{r(\theta_1 + \lambda)} \left[ 1 - e^{-\theta_1(\theta_1 + \lambda)} \right] + \frac{1}{r(\theta_1 + \lambda)(\theta_1 - \theta_2 + \lambda)^2} \left[ e^{-\theta_1(\theta_1 + \lambda)} - e^{-\theta_1(\theta_1 + \lambda)} \right] + \frac{1}{r\theta_1(\theta_1 - \theta_2 + \lambda)^2} \left[ e^{-\theta_1(\theta_1 + \lambda)} - e^{-\theta_1(\theta_1 + \lambda)} \right] + \frac{1}{r(\theta_1 - \theta_2 + \lambda)} \left[ te^{-\theta_1} + \frac{e^{-\theta_1}}{\theta_2^2} - \frac{S\theta_1}{r\theta_2} - \frac{e^{-\theta_1}}{\theta_2^2} \right] 
\]
\[
\frac{d\lambda\theta_1^2}{\mu} \left[ - \frac{e^{-S(\theta_1 + \lambda)}}{r(\theta_1 + \lambda)(\theta_1 - \theta_2 + \lambda)^2} + \frac{e^{-S\theta_1}}{r\theta_2(\theta_1 - \theta_2 + \lambda)^2} + \frac{1}{r(\theta_1 - \theta_2 + \lambda)} \left[ S\theta_1 e^{-\theta_1} + e^{-\theta_1} - \frac{e^{S\theta_1}}{\theta_2^2} \right] \right] = 0
\]

... (1.12)

Any value of $S$ which satisfies equation (1.12) is the optimal $\hat{S}$, and can be derived from the above equation substituting $\frac{dE[C]}{dS} = 0$. Hence given the values of the parameters of the distributions involved and also the holding cost and shortage cost the optimal $S$ namely $\hat{S}$, can be obtained by appropriate computer program.

**EXAMPLES:**

We take $\theta_1 = 1, \theta_2 = 1.5, \lambda = 1, h = 20, d = 30, r = 5, t = 10, \mu = 2.5$ and so $\hat{S} = 3.4524$ The following illustrations are taken up.

**Example (i):**

Changes in $h$ and the other parameters kept fixed.
$\theta_1 = 1, \theta_2 = 1.5, \lambda = 1, d = 30, r = 5, t = 10, \mu = 2.5$

<table>
<thead>
<tr>
<th>$h$</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{S}$</td>
<td>3.4524</td>
<td>3.1538</td>
<td>2.9306</td>
<td>2.7553</td>
<td>2.6128</td>
<td>2.4939</td>
<td>2.3925</td>
</tr>
</tbody>
</table>

Table (1.1)
**Example (ii):**

Changes in $d$ and the other parameters kept fixed.

$\theta_1 = 1.0$, $\theta_2 = 1.5$, $\lambda = 1.0$, $h = 20$, $r = 5$, $t = 10$, $\mu = 2.5$

<table>
<thead>
<tr>
<th>$d$</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{S}$</td>
<td>3.4524</td>
<td>3.6764</td>
<td>3.8829</td>
<td>4.0753</td>
<td>4.2557</td>
<td>4.4261</td>
<td>4.5876</td>
</tr>
</tbody>
</table>

Table (1.2)

![Fig (1.2)](image)

**CASE II**

Let us assume that $t > \frac{S}{r}$.

then $E[C] = h r \int_0^S \frac{S}{r} - \tau g_1(\tau) d\tau + \frac{d}{\mu} \int_0^\tau \tau - \frac{S}{r} g_1(\tau) d\tau + \frac{d}{\mu} \int_0^\tau \tau - \frac{S}{r} \hat{g}_2(\tau) d\tau$

\[\ldots (1.13)\]

Now we have

\[
\frac{dE[C]}{dS} = \frac{h \theta_1}{r(\theta_1 + \lambda)} \left[ 1 - e^{-\frac{S}{r(\theta_1 + \lambda)}} \right] + \frac{d \theta_1}{\mu r(\theta_1 + \lambda)} \left[ e^{-\frac{S}{r(\theta_1 + \lambda)}} - e^{-\frac{S}{r(\theta_1 + \lambda)}} \right] + \frac{d \lambda \theta_2^2}{\mu} \left[ -\frac{e^{-\frac{S}{r(\theta_1 + \lambda)}}}{r(\theta_1 + \lambda)(\theta_1 - \theta_2 + \lambda)^2} + \frac{e^{-\theta_1 t}}{r(\theta_1 - \theta_2 + \lambda)^2} - \frac{1}{r(\theta_1 - \theta_2 + \lambda)} \left( t e^{-\theta_0} + e^{-\theta_0} \right) \right] = 0
\]

\[\ldots (1.14)\]

\[
\therefore \frac{d}{dS} = -\frac{r}{(\theta_1 + \lambda)} \log \left\{ \frac{\mu(\theta_1 + \lambda)}{\theta_1(d + \mu h)} \left[ \frac{h \theta_1}{r(\theta_1 + \lambda)} + \frac{d \theta_1 e^{-\theta_1}}{\mu(\theta_1 + \lambda)} \right] + \frac{d \lambda \theta_2 e^{-\lambda(\theta_1 + \lambda)}}{\mu(\theta_1 + \lambda)(\theta_1 - \theta_2 + \lambda)^2} + \frac{d \lambda \theta_2 e^{-\lambda(\theta_1 + \lambda)}}{\mu(\theta_1 + \lambda)(\theta_1 - \theta_2 + \lambda)^2} \right\}
\]

\[\ldots (1.15)\]
Again the optimal value of $S$ namely $\hat{S}$ is one which satisfies (1.15). It is again derived by an appropriate computer program. We take $\theta_1 = 1$, $\theta_2 = 1.5$, $\lambda = 1$, $h = 20$, $d = 30$, $r = 5$, $t = 10$, $\mu = 2.5$ and so $\hat{S} = 0.2833$. For any value of $\theta_1$ and $\theta_2$, such that $\theta_1 < \theta_2$, optimal $\hat{S}$ is derived in numerical terms. The following illustrations are taken up.

**EXAMPLE (iii):**

Changes in $h$ and the other parameters kept fixed.

\[ \theta_1 = 1, \theta_2 = 1.5, \lambda = 1, h = 20, d = 30, r = 5, t = 10, \mu = 2.5 \]

<table>
<thead>
<tr>
<th>$h$</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{S}$</td>
<td>0.2833</td>
<td>0.2292</td>
<td>0.1924</td>
<td>0.1658</td>
<td>0.1457</td>
<td>0.1299</td>
<td>0.1172</td>
</tr>
</tbody>
</table>

Table (1.3)

![Graph](Fig (1.3))

**EXAMPLE (iv):**

Changes in $d$ and the other parameters kept fixed.

\[ \theta_1 = 1, \theta_2 = 1.5, \lambda = 1, h = 20, r = 5, t = 10, \mu = 2.5 \]

<table>
<thead>
<tr>
<th>$d$</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{S}$</td>
<td>0.2833</td>
<td>0.3276</td>
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<td>0.4558</td>
<td>0.4971</td>
<td>0.5378</td>
</tr>
</tbody>
</table>

Table (1.4)

![Graph](Fig (1.4))
CONCLUSIONS

1. As the value '$h$' increases, the value decreases, thus an increase in the holding cost decreases the size of optimal reserve.

2. As '$d$' which is the shortage cost increases, the model suggests an increase in the optimal value of $\hat{S}$ because the shortage cost is very high and therefore a greater inventory should be maintained which is quite reasonable. This is indicated with corresponding figure.

3. The same kind of phenomena is observed when $\gamma$.

REFERENCES


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