An Active Low-Order Fault-Tolerant State Space Self-Tuner for the Unknown Sample-Data Linear Regular System with an Input-Output Direct Feedthrough Term

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Abstract

A novel active low-order fault-tolerant state space self-tuner for the unknown linear regular system with an input-output direct feedthrough term matrix using observer/Kalman filter identification (OKID) and modified autoregressive moving average with exogenous input (ARMAX) model-based system identification is proposed in this paper. Through OKID, the order determination and a good initial guess of the modified ARMAX model is obtained to improve the performance of the identification process. With the modified adjustable ARMAX-based system identification, a corresponding adaptive digital control scheme is recommended. Besides, by modifying the conventional self-tuning control, a fault tolerant control scheme is also developed for the system. With the detection of fault occurrence, a
quantitative criterion is improved by comparing the innovation process errors estimated by the Kalman filter estimation algorithm. Therefore, a resetting technique of the weighting matrix is amended by adjusting and resetting the covariance matrices of parameter estimation obtained by the Kalman filter estimation algorithm.

**Keywords**: State space self-tuning control; singular system; ARMAX model; fault tolerant control; observer/Kalman filter identification; system identification

1 Introduction

The applications of singular systems (generalized state-space systems and descriptor systems) in economics and demography are well known. Singular systems also appear naturally in describing large scale systems. Over the past decades, many research results concerning about the singular system have successfully solved a great number of complex problems such as impulsive modes [1], stability [2], controllability and observability [3], the sufficient and necessary condition for impulse controllability and observability of time-varying singular systems [4]. However, the tracking issue of the singular system has not been effectively discussed in the literature.

The state-space and self-tuning control methods [5,6] have been shown to be effective in designing advanced adaptive controllers to the linear multivariable stochastic regular system [7]. In those approaches, the standard Kalman state-estimation algorithm [8] has been embedded into an online parameter estimation algorithm. They use state-space self-tuners based on innovation models, where (i) the equivalent internal states can be estimated successively; (ii) the stable/unstable and minimum/non-minimum-phase multivariable systems can be controlled accurately; (iii) the self-tuners are simple, reliable and robust; and (iv) the adaptive Kalman gain can subsequently be obtained.

A singular system can be reformulated into the equivalent regular system model with a direct feedthrough term from the input to the output [1], so one can use the off-line OKID (observer/Kalman filter identification) algorithm [9] to identify the initial parameter, including the direct feedthrough term matrix and order determination. They are used for the online recursive extended-least-squares (RELS) identification based on modified AutoRegressive Moving Average with eXogenous input (ARMAX) model. It will be proposed in this paper, from the input and output data. Then, based on the modified ARMAX model with its corresponding state space novelty form, a novel digital controller design with an input direct feedthrough term matrix to deal with the regular system is suggested in this paper.

One noticed point is that the state-space self-tuning control (STC) scheme for nonlinear stochastic hybrid systems offered by Guo et al. [10] it can estimate the system parameters and can design an adaptive controller based on the estimated parameters at every sampling instant. The framework of the state-space STC seems to agree with that the active fault tolerance in a real-time. As to the faulty
Active low-order fault-tolerant state space self-tuner

4815

system recovery, instead of direct using the estimated covariance matrices, we use
the modified Kalman filter estimation algorithm by utilizing the modified
covariance matrices from estimated errors to improving the parameter estimation
[11]. They are obtained from the RELS algorithm in the conventional STC
scheme for adapting parameter variations. About the faults, abrupt faults and
gradual faults are considered in this paper as well.

The rest of the paper is organized as follows. In Section 2, the novel digital
controller design method with an input direct feedthrough term matrix to the
regular system is presented. Section 3 proposes modified on-line system
identification for the state space and self-tuning control of the unknown linear
regular system with a direct feedthrough term. Section 4 summaries the design
procedure to the modified ARMAX model-based state-space and self-tuner of an
appropriate (low-) order for the unknown linear singular system. In Section 5, a
fault tolerance scheme is proposed by modifying the conventional state-space
self-tuning control approach for the unknown multivariable stochastic singular
system. Finally, an illustrative example is shown in Section 6.

2 Novel digital tracker for linear regular system with a direct
feedthrough term

This section proposes a novel digital controller method for the linear regular
system with a direct feedthrough term matrix. Consider a linear continuous-time
singular system as follows

\[
E_\epsilon \dot{x}_\epsilon(t) = A_\epsilon x_\epsilon(t) + B_\epsilon u_\epsilon(t), \quad (1a)
\]

\[
y_\epsilon(t) = C_\epsilon x_\epsilon(t), \quad (1b)
\]

where \( x_\epsilon(t) \) is the state vector, \( u_\epsilon(t) \) is the control input, and \( y_\epsilon(t) \) is the
output. These constant matrices \( E_\epsilon, A_\epsilon, B_\epsilon, \) and \( C_\epsilon \) all have appropriate
dimensions, and \( E_\epsilon \) is a singular matrix. Reformulate the above singular system
to the equivalent regular system with a direct feedthrough term matrix from input
to output as shown in Appendix B.

\[
\dot{x}_r(t) = A_r x_r(t) + B_r u_r(t), \quad (2a)
\]

\[
y_r(t) = C_r x_r(t) + D_f u_r(t). \quad (2b)
\]

where \( A_r, B_r, C_r, \) and \( D_f \) are the equivalent regular system matrices of
appropriate dimensions. One supposes that the system model is known, but most
real physical system models are unknown. When the system structure and
parameters are unknown, the observer-based linear quadratic digital tracker
(LQDT) cannot be directly designed. To overcome this, the off-line OKID method
is used to determine the appropriate/minimal order state estimator and system
parameters for the unknown sampled-data linear singular system. The
corresponding linear discrete-time system state space innovation form of (1) or (2)
can be formulated through OKID algorithm [9] as follows
\[ x_d(k+1) = Gx_d(k) + Hu_d(k) + Le_o(k), \]  
\[ e_o(k) = y_d(k) - C_sx_o(k) - D_fu_d(k), \]  
where \( y_d(k) \) is the system output at sampling index \( k \), \( L \) is the observer gain, and \( e_o(k) \) is the discrete-time observer error. Here, we would like to point out that the dimensions of the model \( (G, H, C_s, D_f) \) may not the same as those of \( (2) \).

Then, consider a linear discrete-time system as follows
\[ x_d(k+1) = Gx_d(k) + Hu_d(k), \]  
\[ y_o(k) = C_sx_d(k) + D_fu_d(k), \]  
where \( x_d(k) \in \mathbb{R}^n \) is the state vector, \( u_d(k) \in \mathbb{R}^m \) is the control input vector at sampling index \( k \), and \( y_o(k) \in \mathbb{R}^p \) is the measurable output vector. Parameters \( G, H, C_s \) and \( D_f \) are estimated (or given) constant system matrices of appropriate dimensions based on OKID. Furthermore, a typical structure for the sampled-data controlled system illustrated in Fig. 1, where \( x_{ds}(t) \), \( u_d(t) \), and \( y_o(t) \) are the corresponding discrete-time equivalent model state, input, and output, respectively, and \( T \) is the sample time. Based on the identified discrete-time model \( (G, H, C_s, D_f) \), the corresponding continuous-time model \( (A_{ds}, B_{ds}, C_s, D_f) \) can be obtained, where \( A_{ds} = \frac{1}{T} \ln G \) and \( B_{ds} = A_{ds} (G - I_n)^{-1} H \).

\[
\begin{align*}
D_f & \quad E_d \\
\downarrow & \quad + \quad \downarrow \\
\vdots & \quad u_d(kT) \quad T \\
\cdots & \quad \text{Z.O.H.} \\
\downarrow & \quad x_{ds}(t) = A_{ds}x_{ds}(t) + B_{ds}u_d(t) \\
\downarrow & \quad \text{K}_d \\
x_o(kT) & \quad \text{K}_o \\
\downarrow & \quad \text{y}_o(t) \\
\end{align*}
\]

Fig. 1 The equivalent system model of the original analog linear singular system with the digital controller.

The objective here is to derive an optimal state-feedback tracker for the unknown linear regular system with a direct feedthrough term. Define the performance index as
\[
J_d = \frac{1}{2} \sum_{k=0}^{k_0} \left[ C_s x_d(k) + WD_d u_d(k) - r(k) \right]^T \Omega \left[ C_s x_d(k) + WD_d u_d(k) - r(k) \right] + u_d^T(k)Ru_d(k),
\]  
(5)
where $k_f$ is the final sampling index, $Q$ is the positive semi-definite matrix, $R$ is the positive definite matrix, $r(k) \in \mathbb{R}^p$ is the pre-specified reference input vector, and $W$ is a weighting matrix to adjust the controller gain matrix. When the parameters $(Q, R)$ of the performance index have a high-gain property, $(Q, R)$, the linear quadratic analog tracker (LQAT) yields a better tracking performance and high control input. It is well-known that the high-gain controller/observer induces a high quality performance on trajectory tracking design/state estimation, and it can also suppress system uncertainties, such as perturbations, parameter variations, modeling errors and external disturbances. For these reasons, the controller and observer with a high-gain property is adopted in our approaches. However, the high-gain property of the analog tracker usually yields large control signals, which might cause the system actuator to saturate and give unsatisfactory system response. To overcome this difficulty, the tracker is redesigned based on the advanced digital redesign technique equipped with a suitably large sampling period and zero order hold, which yields an equivalent digital controller but with a low gain, without possibly losing the high quality performance. However, a large sampling period usually induces a degradation of the tracking performance. Therefore, in general, a suitable compromise between the pre-specified performance and the selections of the sampling time $T_s$, weighting matrices $W, (Q, R)$ in (5) and $(Q_s, R_s)$ in (40) should be considered.

Notice that in Fig. 1, if the current control input $u_d(k)$ at time instant $k$ is implemented by the current state $x_d(k)$ and the current reference input $r(k)$, by which the current output $y_d(k)$ exists already, then the current control $u_d(k)$ can not force the current output $y_d(k)$ to track the current trajectory $r(k)$. Therefore, the $u_d(k)$ is to be determined based on the design goal $y_d(k+1) = y(k+1) = r(k+1)$ for $k = 0, 1, 2, \ldots$, which involves the predicted output and the predicted reference input, but not the current output and the current reference input. The above design goal implies $y_d(k) = y(k) = r(k)$ for $k = 1, 2, 3, \cdots$ also. As a result, for tracking purpose, a prediction-based digital redesign method [12] is proposed to achieve the design goal. Due to this fact the control law $u(k) = -K_d x(k) + E_d r^*(k)$, where $r^*(k) = r(k+1)$, is to be determined as follows.

In order to simplify the notations, denote $C, \tilde{D}, x(k)$, and $u(k)$ as $C$, $WD$, $x_d(k)$ and $u_d(k)$, respectively. Then, one has the state equation

$$x(k+1) = Gx(k) + Hu(k),$$

(6)
and the co-state equation [13]
\[ \dot{\lambda}(k) = G^T \dot{\lambda}(k+1) + C^T Q(Cx(k) + \tilde{D}u(k) - r^*(k)), \] (7)
where \( k \leq k_f \), \( r^*(k) = r(k+1) \), and \( \lambda \) is the Lagrange multiplier with the stationary condition
\[ 0 = H^T \dot{\lambda}(k+1) + \tilde{D}^T Q(Cx(k) + \tilde{D}u(k) - r^*(k)) + Ru(k), \]
or
\[ u(k) = -(R + \tilde{D}^T Q\tilde{D})^{-1}(H^T \dot{\lambda}(k+1) + \tilde{D}^T Q(Cx(k) - r^*(k))), \] (8)
and the boundary condition
\[ \dot{\lambda}(k_f) = C^T p(k)[Cx(k_f) + \tilde{D}u(k_f) - r^*(k_f)]. \] (9)
Assume \( \lambda(k) \) can be written as the following form
\[ \lambda(k) = P(k)x(k) - V(k). \] (10)
Substituting (10) into (8) yields
\[ u(k) = -(R + \tilde{D}^T Q\tilde{D})^{-1}(H^T \dot{\lambda}(k+1) + \tilde{D}^T Q(Cx(k) - r^*(k))), \]
and the boundary condition
\[ \dot{\lambda}(k_f) = C^T p(k)[Cx(k_f) + \tilde{D}u(k_f) - r^*(k_f)]. \]
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\[ u(k) = -(R + \tilde{D}^T Q\tilde{D})^{-1}(H^T \dot{\lambda}(k+1) + \tilde{D}^T Q(Cx(k) - r^*(k))), \]
and the boundary condition
\[ \dot{\lambda}(k_f) = C^T p(k)[Cx(k_f) + \tilde{D}u(k_f) - r^*(k_f)]. \] (9)
Equation (7) can be rewritten as
\[ \lambda(k) = G^T \dot{\lambda}(k+1) + C^T QCx(k) + C^T Q(\tilde{D}u(k) - r^*(k)). \] (13)
Similarly, substituting (12) into (13) yields
\[ \lambda(k) = G^T P(k+1)(Gx(k) + Hu(k)) - G^T V(k+1) + C^T QCx(k) + C^T Q(\tilde{D}u(k) - r^*(k)). \] (14)
And substituting (11) into (14) yields
\[ \lambda(k) = G^T P(k+1)H(\bar{R} + H^T P(k+1)H)^{-1}H^T V(k+1) + C^T Q\bar{D}(\bar{R} + H^T P(k+1)H)H^T V(k+1) \]
\[ -G^T V(k+1) - G^T P(k+1)H(\bar{R} + H^T P(k+1)H)^{-1}(H^T P(k+1)G + \bar{D}^T QC)x(k) \]
\[ -C^T \bar{Q}(\bar{R} + H^T P(k+1)H)^{-1}(H^T P(k+1)G + \bar{D}^T QC)x(k) + C^T QCx(k) \]
\[ + G^T P(k+1)x(k) + C^T \bar{Q}(\bar{R} + H^T P(k+1)H)\bar{D}^T Qr^*(k) \]
\[ + G^T P(k+1)H(\bar{R} + H^T P(k+1)H)^{-1}\bar{D}^T Qr^*(k) - C^T Qr^*(k). \] (15)
Combining (15) and (10) yields
\[ P(k)x(k) - G^T P(k+1)x(k) + G^T P(k+1)H(\bar{R} + H^T P(k+1)H)^{-1}(H^T P(k+1)G + N^T)x(k) \]
\[ + N(\bar{R} + H^T P(k+1)H)^{-1}(H^T P(k+1)G + N^T)x(k) - C^T QCx(k) \]
\[ = V(k) - G^T V(k+1) + (G^T P(k+1)H + N)(\bar{R} + H^T P(k+1)H)^{-1}H^T V(k+1) \]
\[ + (G^T P(k+1)H + N)(\bar{R} + H^T P(k+1)H)^{-1}\bar{D}^T Qr^*(k) - C^T Qr^*(k). \] (16)
Rewriting the previous equation (16), we have the steady-state solutions
\[ V(k+1) = V(k) = V, \]
Active low-order fault-tolerant state space self-tuner

\[ P(k+1) = P(k) = P, \]
\[ P = G^T PG + C^T QC - (G^T PH + N)(\bar{R} + H^T PH)^{-1}(H^T PG + N^T), \]  \hspace{1cm} (17)

and
\[ V = G^T V + C^T Qr^*(k) - (G^T PH + N)(\bar{R} + H^T PH)^{-1}(H^T V + \bar{D}^T Qr^*(k)). \]  \hspace{1cm} (18)

From (17), \( K_d \) can be formulated as
\[ K_d = (\bar{R} + H^T PH)^{-1}(H^T PG + N^T). \]  \hspace{1cm} (19)

Then, from (18) and (19), one has
\[ V = G^T V + C^T Qr^*(k) - K_d^T (H^T V + \bar{D}^T Qr^*(k)). \]  \hspace{1cm} (20)

Rearranging the above equation (20) yields
\[ V = -(I - (G - HK_d)^T)^{-1}(\bar{D}K_d - C)^T Qr^*(k). \]  \hspace{1cm} (21)

Finally, by substituting (21) into (11), the optimal control law is given as follows
\[ u(k) = -K_d\dot{x}(k) + E_d r^*(k), \]  \hspace{1cm} (22)

where
\[ K_d = \left[ \bar{R} + H^T PH \right]^{-1} \left[ H^T PG + N^T \right], \]  \hspace{1cm} (23)
\[ E_d = \left[ \bar{R} + H^T PH \right]^{-1} \left\{ -H^T \left[ I - (G - HK_d)^T \right]^{-1} \left[ \bar{D}K_d - C \right]^T + \bar{D}^T \right\} Q, \]  \hspace{1cm} (24)

and \( r^*(k) = r(k+1) \) for the tracking purpose [12], where \( P \) is the positive definite and symmetric solution of the following Riccati equation
\[ P = G^T PG + C^T QC - (G^T PH + N)(\bar{R} + H^T PH)^{-1}(H^T PG + N^T). \]  \hspace{1cm} (25)

Then, a novel digital controller gains \((K_d, E_d)\) is given for the equivalent linear regular system with a direct feedthrough term.

3 Modified ARMAX model-based state-space self-tuning control for the linear regular system with a direct feedthrough term

The structure of the state-space STC scheme includes a parameter and state estimator and a controller design. A typical state-space STC structure is illustrated in Fig. 2.

![Fig. 2 Block diagram of a typical state-space self-tuner control.](image-url)
Under this framework, the parameters and states of the unknown model are estimated from the control inputs $u_d(kT), u_d((k-1)T), u_d((k-2)T), \cdots$ and the system outputs $y_d(kT), y_d((k-1)T), y_d((k-2)T), \cdots$, where $u_d(kT)$ based on some prediction concept, is to be estimated. Depend on the estimated parameters $\theta(k)$ of modified ARMAX model, an appropriate controller (22) can be designed by the corresponding state-space model described in Section 3.4. Then, the reference input $r^*(k)$ and the designed adaptive controller generate real-time control actions for the unknown dynamic system. The detail of the algorithm to estimate the system parameters $\theta(k)$ is described in Section 3.4. And the process cycle is repeated until the control goal is achieved. Notice that if the control input is persistently excited, the convergence to the true system parameters is guaranteed by Ljung [14].

3.1 Modified ARMAX model for self-tuning control scheme

In this section, one proposes the modified ARMAX model for the regular system with a direct feedthrough term and reviews some preliminary structures useful for the hybrid state-space self-tuning control law design. The class of random disturbances is characterized by the fact that their spectral density function has a frequency dependence that can be approximated by a rational frequency function (i.e., a ratio of polynomials functions with real coefficients) [15,16]. The majority of random disturbances occurring in automatic control systems may be practically and accurately described as a Gaussian discrete-time white noise passed through a filter. The knowledge of this filter, called the disturbance model, allows the disturbance to be completely characterized (with the approximation of a scaling factor). For the design of digital controller in the presence of random disturbances, one considers a joint plant and disturbance model called the ARMAX model. In this paper, we consider the multivariable system with the results then extendable to the general multivariable case as well.

The disturbed plant and the modified ARMAX model take the following forms, respectively

$$G_i(z^{-1}) y(k) = H_i(z^{-1}) u(k) + \xi(k), \quad (26a)$$

and

$$G_i(z^{-1}) \tilde{y}(k) = H_i(z^{-1}) u(k) + D_i(z^{-1}) e(k), \quad (26b)$$

where $y(k) \in \mathbb{R}^p$ is the plant output, $u(k) \in \mathbb{R}^m$ is the plant input, $\xi(k)$ is the disturbance of system, and $\tilde{y}(k)$ is the estimated value of $y(k)$.

$$\tilde{y}(k) = \begin{bmatrix} \tilde{y}_1(k) & \tilde{y}_2(k) & \cdots & \tilde{y}_p(k) \end{bmatrix}^T,$$

$$u(k) = \begin{bmatrix} u_1(k) & u_2(k) & \cdots & u_m(k) \end{bmatrix}^T,$$

$$e(k) = \begin{bmatrix} e_1(k) & e_2(k) & \cdots & e_p(k) \end{bmatrix}^T,$$
\[ \xi(k) = D_1(z^{-1})e(k), \]
\[ G_i(z^{-1}) = I_p + G_{o3}z^{-1} + \cdots + G_{oq}z^{-q}, \quad G_{oi}(i = 1, 2, \ldots, q) \in \mathbb{R}^{p \times p}, \]
\[ H_i(z^{-1}) = H_{o0} + H_{o1}z^{-1} + \cdots + H_{oq}z^{-q}, \quad H_{oi}(i = 0, 1, \ldots, q) \in \mathbb{R}^{p \times m}, \]
\[ D_i(z^{-1}) = I_p + D_{o3}z^{-1} + \cdots + D_{oq}z^{-q}, \quad D_{oi}(i = 0, 1, \ldots, q) \in \mathbb{R}^{p \times p}. \]

Here, the special characteristic of the modified ARMAX model is that it involves the \( H_{o0} \) matrix. Traditional ARMAX model does not have the \( H_{o0} \) matrix, because the system does not have the direct transmission term. The proposed ARMAX model has the \( H_{o0} \) matrix, so it fits the regular system with a direct feedthrough matrix.

An alternative representation of the modified ARMAX model (26b) is given by

\[ \tilde{y}(k) = G_i^{-1}(z^{-1})H_i(z^{-1})u(k) + G_i^{-1}(z^{-1})D_1(z^{-1})e(k). \] (27)

Equation (27) can be organized as follows

\[ \tilde{y}(k) = G_{oi}(k-1) + \cdots + G_{op}(k-n_o) + H_{oi}u(k) + \cdots + H_{om}u(k-n_m) + D_{oi}e(k-1) + \cdots + D_{op}e(k-n_p), \] (28a)

where \( \tilde{y}(k) = [\tilde{y}_1(k), \tilde{y}_2(k), \ldots, \tilde{y}_p(k)]^T \), \( u(k) = -K_j(k-1)x_j(k) + E_j(k-1)r^*(k) \) is the prediction of the control input, \( u(k) \) and \( y(k) \) denote the input and output at time indices \( k \) (\( k = 0, 1, \ldots \)). Notation \( e(k) \) is the residual vector, \( n_{oy}, n_{uu}, n_{ee} \) are the orders of \( y, u, e \), respectively, determined by the off-line OKID method. For tracking purpose, their orders are supposed to be the same, i.e. \( n_{oy} = n_{uu} = n_{ee} \). Equation (28a) can be rewritten as

\[ \tilde{y}_i(k) = \sum_{j=1}^{num} \theta_j(k-1)\phi_j(k) = \theta_i^T(k-1)\phi(k), \quad \text{for } i = 1, 2, \cdots, p, \] (28b)

where \( num = (3q+1)p \) is the amount of the components in

\[ \phi(k) = [\gamma_1(k-1), \gamma_2(k-2), \ldots, \gamma_{n_y}(k-n_y), u(k-1), \ldots, u(k-n_u), e(k-1), \ldots, e(k-n_e)]^T \]

for \( p = m \) in (28b), \( \theta_j(k) \) denotes the \( j \)-th parameter of modified ARMAX model to be estimated for the \( i \)-th estimated output \( \tilde{y}_i(k) \). Thus, each estimated output \( \tilde{y}_i(k) \) is identified from each class of \( \phi(k) \) as

\[ \tilde{y}_i(k) = \theta_i^T(k-1)\phi(k), \quad i = 1, 2, \cdots, p, \] (29a)

and the standard RELS algorithm is applied by

\[ \tilde{\theta}_i(k) = \tilde{\theta}_i(k-1) + \frac{S_i(k-1)\phi(k)}{\lambda(k) + \phi^T(k)S_i(k-1)\phi(k)}e_i(k), \] (29b)

\[ S_i(k) = \frac{1}{\lambda(k)} \left[ S_i(k-1) - \frac{S_i(k-1)\phi(k)\phi(k)^T S_i(k-1)}{\lambda(k) + \phi^T(k)S_i(k-1)\phi(k)} \right], \] (29c)

where \( \lambda(k) \) is the forgetting function to discount the old measurements, and can be determined by the first-order difference equation, \( \lambda(k) = \lambda_0\lambda(k-1) + (1 - \lambda_0) \), with the initial condition \( 0 < \lambda(0) < 1 \), and the updating factor \( 0 < \lambda_0 < 1 \). Also,
$S_i(k) \in \mathbb{R}^{(n_{num}) \times (n_{num})}$ is the parameters estimation error covariance matrix with $S_i(0) = \alpha_i I_{(n_{num}) \times (n_{num})}$, where $\alpha_i$ is the positive scalar, and the residual vector of each output is given by

$$\varepsilon_i(k) = y_i(k) - \theta_i^T(k-1)\phi_i(k).$$

### 3.2 Preliminary of system identification

In this paper, one slightly modifies the basic structure of discrete-time state-space model, so that it is useful for the hybrid state-space self-tuning control law design for the linear singular system. Once having the estimated parameters $\theta_i(k)$ from the modified RELS algorithm, the modified ARMAX model for the STC scheme can accurately approximate the state of the linear singular system. Since the initial parameter $\theta_i(0)$ of the (modified) ARMAX model significantly affects the convergent speed of the (modified) RELS process, in order to get suitable initial parameter $\theta_i(0)$ to improve the performance of on-line RELS, we apply the off-line OKID to evaluate it here.

Here, we would like to point out that no matter the on-line RELS or the off-line OKID; they are desired to identify the parameters of the state-space innovation form for the open-loop system, not the closed-loop system. Consequently, for the off-line OKID, the control input $u_i(k)$ in the following regressive is available for the time instant $t = kT$, i.e., the predicted control input $u_i^*(k)$ is not required. As for the detail process of OKID, it is given in Section 3.4.

The regressive $\phi_i(k)$ in $\tilde{y}_i(k) = \theta_i^T(k-1)\phi_i(k)$ is composed of

$$-y_1(k-1) \ldots y_1(k-n_ay),$$

$$-y_2(k-1) \ldots y_2(k-n_y), \ldots, -y_p(k-1) \ldots y_p(k-n_p), u_1(k) \ldots u_1(k-n_u), u_2(k) \ldots u_2(k-n_u), \ldots, u_m(k) \ldots u_m(k-n_m), \varepsilon_1(k-1) \ldots \varepsilon_1(k-n_\varepsilon), \varepsilon_2(k-1) \ldots \varepsilon_2(k-n_\varepsilon), \ldots, \varepsilon_p(k-1) \ldots \varepsilon_p(k-n_\varepsilon).$$

### 3.3 State-space innovation model

A preliminary structure of the discrete state-space observer of the linear system is presented in this section [7]. Consider the following linear discrete stochastic system as

$$x(k+1) = Gx(k) + Hu(k) + w(k),$$

$$y(k) = Cx(k) + Du(k) + v(k),$$

where $G \in \mathbb{R}^{n_x \times n_x}$, $H \in \mathbb{R}^{n_x \times m}$, $C \in \mathbb{R}^{p \times n_x}$, and $D \in \mathbb{R}^{p \times m}$ are system, input, output matrices, and direct transmission matrix, respectively, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are state, input, and output vectors, respectively, $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ are zero-mean white noise sequences with covariance matrices as

$$E \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w^T(l) \\ v^T(l) \end{bmatrix} = \begin{bmatrix} Q & S \\ S & R \end{bmatrix} \delta_{k,l},$$

(32)
Q ≥ 0, R > 0, δ_{k,l} = 1 if k = l, and δ_{k,l} = 0 if k ≠ l, k,l = 0,1,2,... . System (31) can be transferred into the block observable form, if the rank of the following observability matrix
\[ \Theta = \left[ (CG^{-1})^T, (CG^{-2})^T, \ldots, (CG)^T, C^T \right]^T \] (33)
is equal to n.

Note that the observability index of Θ is r = n/p, if it is an integer (otherwise, it is undefined). This constraint means that the Kronecker indices of system (31) are all such integer r that satisfies n = rp. When system (31) is blocks observable, it can be transformed into the block observable companion form as follows
\[ x_o(k+1) = G_o x_o(k) + H_o u(k) + w_o(k), \] (34a)
\[ y(k) = C_o x_o(k) + D_o u(k) + v_o(k). \] (34b)
System (34) can be represented by a state-space innovation model [17] as
\[ x_o(k+1|k) = G_o x_o(k|k-1) + H_o u(k) + K_o(k)e_o(k), \] (35a)
\[ y(k) = C_o x_o(k|k-1) + D_o u(k) + e_o(k), \] (35b)
where K_o(k) is the Kalman gain. For the special case D_o = O_p, K_o(k) can be computed by the following algorithm [18]
\[ K_o(k) = \left[ G_o P_o(k) C_o^T + S_o \right] \left[ C_o P_o(k) C_o^T + R_o \right]^{-1}, \] (36a)
\[ P_o(k+1) = [G_o - K_o(k) C_o] P_o(k) \left[ G_o - K_o(k) C_o \right]^T + K_o(k) R_o K_o^T(k) - S_o K_o^T(k) - K_o(k) S_o^T + Q_o \]
\[ = E \{ \hat{x}_o(k+1|k) \hat{x}_o(k+1|k)^T \}, \] (36b)
\[ P_o(0) = E \{ [x_o(0) - \hat{x}_o(0)] [x_o(0) - \hat{x}_o(0)]^T \}, \] (36c)
in which \( \hat{x}_o(k+1|k) \) is the optimal estimate of \( x_o(k+1) \) by the measurement data up to \( y(k) \), i.e., \( y(i) \) for \( i = 0,1,\cdots,k, \( \hat{x}_o(k|k-1) = x_o(k) - \hat{x}_o(k|k-1) \) is the estimate error, \( e_o(k) = y(k) - C_o x_o(k|k-1) = C_o \hat{x}_o(k) + v_o(k) \) is the zero-mean white noise sequence with \( R_o = E \{ e_o(k) e_o^T(k) \} = C_o P_o(k) C_o^T + R_o \), and \( e_o(k) \) is called the innovation process. However, for the case \( D_o \neq O_p \), no literature investigates this problem yet.

Here, we would like to point out that the restriction for dimensions of a state and output vectors, i.e. \( r = n/p \) is an integer, does not exist in the practical implementation, since the system itself with the state number n is supposed to be unknown, its corresponding mathematical model is determined and implemented by the off-line OKID [9], where the determined system model with an integer ratio of state number to output number can always be satisfied.

### 3.4 Initialization for fast on-line system identification and observer for the unknown linear regular system with a direct feedthrough term

#### 3.4.1 Modified ARMAX model and state-space innovation form
By the proposed modified ARMAX model (26) for the self-tuning control, the
discrete-time state-space innovation model (35) is constructed to design the
control input \( u_d(k) \) for the unknown real system.

In this paper, we select the class of modified ARMAX model with \( m \)-inputs
and \( p \)-outputs for \( p=m \) in (28b). An alternative representation of the modified
ARMAX model (27) is given by
\[
\tilde{y}(k) = G_l^{-1}(z^{-1})H_l(z^{-1})u(k) + G_l^{-1}(z^{-1})D_l(z^{-1})e(k),
\]
in which (37) is in the left matrix fraction description form (LMFD) [7]. The first
and second terms in the right-hand side of (37) share the same left characteristic
matrix polynomial \( G_l^{-1}(z^{-1}) \), which represents the effects of the control and the
disturbances. Once \( G_l^{-1}(z^{-1}) \) has been specified to characterize the dynamics of
the plant, the residual vector model \( G_l^{-1}(z^{-1})D_l(z^{-1})e(k) \) presents an adjustable
moving average process of the noise input \( D_l(z^{-1})e(k) \). A system in an observable
block companion form can be represented in the state-space innovation form
[19-22] as
\[
\hat{x}_o(k+1) = G_o\hat{x}_o(k) + H_o u(k) + K_o e_o(k),
\]
\[
e_o(k) = y(k) - C_o\hat{x}_o(k) - D_o u(k),
\]
where \( \hat{x}_o(k+1) \) is the estimation of system state \( x(k) \) in the observer coordinates, and the initial state
is given as \( \hat{x}_o(0) = \hat{C}^\dagger y(0), \) where \( C \) is the pseudo-inverse of matrix \( C \) and
\( e_o(k) = y(k) - C_o\hat{x}_o(k) - D_o u(k). \)

However, the zeros of \( D_l(z^{-1}) \) in (37) may not be all in the unit circle, so
that the eigenvalues of the observer gain \( K_o \) in (38) may not all lie in the unit
circle either. Instead of the Kalman gain \( K_o \), one could design the digital estimator
gain \( L_o \) to replace \( K_o \). For the online self-tuning control, the digital estimator
gain is indirectly designed via the discrete-time observer design based on (39).
Therefore, the closed-loop estimator matrix \( G_o(k) - L_o(k)C_o \) has all its
eigenvalues strictly lying inside the unit circle.

The observer gain is given as

\[ L_o(k) = \left( (C_o P(k) C_o^T + R_o)^{-1} C_o P(k) G_o^T(k) \right)^T, \]  

where \( P(k) \) is the solution of the Riccati equation

\[ G_o(k) P(k) G_o^T(k) - P(k) - (G_o(k) P(k) C_o^T)(C_o P(k) C_o^T + R_o)^{-1}(C_o P(k) G_o^T(k)) + Q_o = 0 \]

in which weighting matrices \( Q_o \geq 0 \) and \( R_o > 0 \) with appropriate dimensions. Then, the corresponding state-space innovation form of (38) becomes

\[ \hat{x}_o(k+1) = G_o(k) \hat{x}_o(k) + H_o(k) u(k) + L_o(k) e_o(k), \]

\[ e_o(k) = y(k) - C_o \hat{x}_o(k) - D_o u(k). \]

### 3.4.2 The initial parameters of ARMAX model based on OKID

The initial parameters \( \theta_i(0) \) of the modified ARMAX model significantly affect the convergent speed of RELS process. In order to increase the convergent speed of RELS algorithm, we predict the initial parameters \( \theta_i(0) \) of the modified ARMAX model by OKID. For getting the initial parameters \( \theta_i(0) \) of RELS algorithm \( (29) \), we perform the off-line system identification scheme OKID [9] to obtain the discrete system realization \( G, H, C, D \) and \( F \) firstly. Then, transfer them \( (G, H, C, D, F) \) into the corresponding observer form \( (G_o, H_o, C_o, D_o, F_o) \) in (38) by

\[ \Theta = \left[ (CG^{-1})^T, (CG^T)^T \right]^T, \]

\[ T_o = [G^{-1} T_{o1}, G^{-2} T_{o1}, \cdots, G T_{o1}, T_{o1}], \quad T_{o1} = \Theta^{-1} C_o^T, \quad C_o = C T_o = [I_p, 0_p, \cdots, 0_p], \]

\[ D_o = H_o 0_p, \quad G_o = T_o^{-1} G T_{o1}, \quad H_o = T_o^{-1} H = [H_{o1}^T, H_{o2}^T, \cdots, H_{o p}^T]^T, \quad F_o = T_o^{-1} F. \]

Based on (37)-(38), we have the modified ARMAX model as

\[ y(k) + G_{o11} y(k-1) + G_{o21} y(k-2) + \cdots + G_{o p1} y(k-p) \]

\[ = H_{o1} u^*(k) + H_{o2} u(k-1) + \cdots + H_{o p} u(k-p) + D_{o1} \epsilon(k-1) + D_{o2} \epsilon(k-2) + \cdots + D_{o p} \epsilon(k-p), \]

where

\[ G_o = \begin{bmatrix} G_{o11} & \cdots & G_{o1p} \\ \vdots & \ddots & \vdots \\ G_{o p1} & \cdots & G_{o p p} \end{bmatrix}, \quad H_o = \begin{bmatrix} H_{o11} & \cdots & H_{o1 m} \\ \vdots & \ddots & \vdots \\ H_{o p1} & \cdots & H_{o p m} \end{bmatrix}, \quad D_o = \begin{bmatrix} D_{o11} & \cdots & D_{o1 p} \\ \vdots & \ddots & \vdots \\ D_{o p1} & \cdots & D_{o p p} \end{bmatrix}, \]

and \( i = 1, 2, \cdots, p \), then one has the coefficient matrices \( (G_o, H_o, D_o = G_o + F_o) \) of the modified ARMAX model in (37). Thus, the above parameters of modified ARMAX model (42) can be utilized as the initial parameters \( \theta_i(0) \) of RELS method (29) in STC.
Consider the class of unknown continuous-time linear stochastic singular systems as follows:

\[ \dot{x}(t) = Ax(t) + Bu(t) + w'(t), \quad y(t) = Cx(t) + v'(t), \]

where \( A : \mathbb{R}^n \to \mathbb{R}^n, \) \( B : \mathbb{R}^m \to \mathbb{R}^{nxm}, \) \( C : \mathbb{R}^n \to \mathbb{R}^p, \) \( u(k) \in \mathbb{R}^m \) is the control input, \( x(k) \in \mathbb{R}^n \) is the state vector, \( y(t) \in \mathbb{R}^p \) is the measurable output vector, \( w'(t) \) and \( v'(t) \) are uncorrelated white noise processes. It is assumed that \( A, B, C, \) \( w'(t), \) and \( v'(t) \) with appropriate dimensions are unknown. In the following, we show the design procedure of class of multi-input multi-output (MIMO) model in (42); however the results can be extended to the general multivariable case for \( \rho > 2. \) The structure of the proposed state-space self-tuner is shown in Fig. 3.

Fig. 3 Structure of the hybrid state-space self-tuner with a modified ARMAX model with order and initial parameter determinations by the off-line OKID for the unknown sample-data linear regular system with a direct feedthrough term.

The design procedure is given as follows:

Step 1) For the unknown continuous-time linear singular system (43), choose...
an appropriate modified ARMAX model (27) to be used to identify this system.

i) Perform the off-line OKID scheme [9] to obtain system and observer-gain Markov parameters of the OKID model, then use the eigensystem realization algorithm (ERA) method to decide an appropriate low-order ARMAX model and obtain the discrete system realization $G_o, H_o, C_o, D_o$, and $F_o$. Finally, determine the appropriate weighting matrix $W$ in (5) via the off-line approach shown in Remark 1.

ii) Based on the state-space innovation form (41) and the modified ARMAX model, initial parameter $\theta(0)=[G_{o1}(0) \ G_{o2}(0) \ H_{o1}(0) \ H_{o2}(0) \ D_{o1}(0) \ D_{o2}(0)]^T$ of the modified ARMAX model can be reversely obtained by $G_o(0)$, $H_o(0)$, $D_o(0)$ and $F_o(0)$, where $H_{o1}(0)=D_o(0)$, $D_{o1}(0)=G_{o1}(0)+F_{o1}(0)$, and $D_{o2}(0)=G_{o2}(0)+F_{o2}(0)$.

Step 2) When the modified ARMAX model is chosen, perform the parameter identification at each sampling period $T$.

i) Set some reasonable initial parameters to perform the state-space RELS algorithm (29). Let the number of $\theta$ be $\theta_{num}$. Also, set $S(0)=\alpha I_{\theta_{num} \times \theta_{num}} > 0$, $0 < \lambda_0 < 1$, $0.9 < \lambda(0) < 1$, $\hat{x}(0) = C^t y_o(0)$, and the initial coefficient matrix $\theta(0)$ which is obtained by OKID in Step 1.

ii) Predict the control input $u_o^*(0)$ for the on-line system identification as $u_o^*(0) = -K_d(0)x_o(0) + E_d(0)r(1)$, where the $K_d(0)$ and $E_d(0)$ are obtained by (23) and (24), and $\hat{x}(0) = C^t y(0)$.

iii) For on-line identifying the given continuous-time linear singular system (43) with piecewise-constant control input, one utilizes the information of input and output to determine the updated parameters $\theta(k)$ at each sampling interval $T$ by RELS algorithm, where the prediction control input $u_o^*(k)$ for the on-line system identification is determined by $u_o^*(k) = -K_d(k-1)x_o(k) + E_d(k-1)r(k+1)$.

Step 3) Estimate states at each sampling period $T$:

Based on the estimated parameter in the modified ARMAX model, estimate the predicted state $\hat{x}_o(k+1|k)$ in (44). Select appropriate $\{Q_o, R_o\}$ in (40) to have the high-gain property digital estimator gain (39). The associated state-space observer (41), for instance $\rho = 2$, is given by
\[ \hat{x}_o(k+1|k) = G_o(k)\hat{x}_o(k|k-1) + H_o(k)u(k) + L_o(k)e_o(k), \]
\[ y_o(k|k-1) = C_o\hat{x}_o(k|k-1) + D_o u_d(k), \]

where \( u(k) = -K_o(k)\hat{x}_o(k) + E_o(k)v(k+1), \)
\[ e_o(k) = y(k) - C_o\hat{x}_o(k|k-1) - D_o u_d^*(k), \]
\[ G_o(k) = \begin{bmatrix} -G_o1(k) & I_p \\ -G_o2(k) & 0_p \end{bmatrix} \in \mathbb{R}^{2p \times 2p}, \quad H_o(k) = \begin{bmatrix} H_{o1}(k) \\ H_{o2}(k) \end{bmatrix} \in \mathbb{R}^{2p \times p}, \]
\[ D_o(k) = H_{o3}(k) \in \mathbb{R}^{p \times p}, \quad C_o = [I_p, \quad 0_p] \in \mathbb{R}^{p \times 2p}, \quad \hat{x}_o(k|k-1) \in \mathbb{R}^{2p}, \]
\[ L_o(k) = ((C_o(k)P(k)C_o^T(k) + R_o)^{-1}C_o(k)P(k)G_o^T(k))^{T} \in \mathbb{R}^{2p \times p}, \]

where \( P(k) \) is the solution of the Riccati equation
\[ G_o(k)P(k)G_o^T(k) - P(k) - (G_o(k)P(k)C_o^T(k) + R_o)^{-1}(C_o(k)P(k)G_o^T(k))^{T} + Q = 0. \]

Step 4) Generate the digital control input at each sampling period \( T \):

i) Select appropriate weighting matrices \( \{Q, R\} \) in (25) to have the high-gain property digital control law in (22)-(24).

ii) Compute the digital control gains \( K_d(k) \) and \( E_d(k) \), by the digital control formula in (23)-(24) as follows:
\[ \tilde{D}(k) = WD_o(k) \quad \text{and} \quad W \quad \text{is a weighting matrix to adjust the controller gain matrix} \]
\[ \tilde{R}(k) = R + \tilde{D}^T(k)Q\tilde{D}(k), \]
\[ N(k) = C_o^TQ\tilde{D}(k), \]
\[ K_d(k) = \begin{bmatrix} \tilde{R}(k) + H_o^T(k)P(k)H_o(k) \end{bmatrix}^{-1} \begin{bmatrix} H_o^T(k)P(k)G_o(k) + N^T(k) \end{bmatrix}, \]
\[ E_d(k) = \begin{bmatrix} -H_o^T(k) \left[ I - (G_o(k) - H_o(k)K_d(k))^{T} \right]^{-1} (\tilde{D}(k)K_o(k) - C(k))^{T} + \tilde{D}^T(k) \end{bmatrix} \times \begin{bmatrix} \tilde{R}(k) + H_o^T(k)P(k)H_o(k) \end{bmatrix}^{-1} Q, \]

where
\[ P(k) = G_o^T(k)P(k)G_o(k) + C_o^T(k)QC_o(k) - (G_o^T(k)P(k)H_o(k) + N(k)) \times (\tilde{R}(k) + H_o^T(k)P(k)H_o(k))^{-1} (H_o^T(k)P(k)G_o(k) + N^T(k)), \]
in which
\[ P(k+1) = P(k). \]

iii) Set \( k = k+1 \). Go to Step 2-(iii) and continue the adaptive control process.

Remark 1: Since the off-line identification algorithm can have sufficiently large amount of input-output mapping, it yields a better performance on the system identification than the on-line based identification algorithm. The weighting matrix \( W \) in (5) for the off-line based Algorithm 1
(by quadratic suboptimal tracker, proposed in Section 2) is then set to be 1 in general. As for the on-line based Algorithm 2 (by ARMAX model state-space self-tuning discrete linear quadratic tracker, proposed in Section 3), to resolve this problem, it’s suggested to set the testing reference as \( \tilde{r}(kT) \) for \( k = 0,1,2,\cdots,N \), then apply the above design procedure to select the appropriate weight matrix \( W \) from a specified certain range \( W^T = \begin{bmatrix} W & \overline{W} \end{bmatrix} \), where \( W \) and \( \overline{W} \) denote the upper and low bounds of \( W^T \), through some off-line sophistical methodologies, such as genetic algorithm or evolutionary programming (EP). Although the on-line based Algorithm 2 is much complicating than the off-line based Algorithm 1, it can deal with the fault-tolerant control; however, the off-line based Algorithm 1 does not.

5 Self-tuning control with fault tolerance

5.1 Problem statement

Consider the class of continuous time linear singular systems. If the system states or inputs are in partial faults, the system dynamics can be represented by

\[
Ex(t) = Ax(t) + Bu(t) + \sum_{\nu=1}^{\sigma} \beta_{\nu}(t - \tau_{\nu}) \left[ Ax(t) + Bu(t) \right] + w(t),
\]

where \( \sum_{\nu=1}^{\sigma} \beta_{\nu}(t - \tau_{\nu}) \left[ Ax(t) + Bu(t) \right] \) represents the dynamic changes caused by the unknown and unanticipated faulty modes of states or inputs. Two typical faults, gradual faults and abrupt faults are considered on-line. Their characteristics are described by the time-varying function, \( \beta_{\nu}(\cdot) \left[ Ax(t) + Bu(t) \right] \) \cite{23}; \( \beta_{\nu}(\cdot) \) and \( \tau_{\nu} \) are unknown due to the possible occurrence of unanticipated faults. If \( \sigma = 1 \), system (45) has a single fault. If \( \sigma = 2,3,4,\cdots \), it means the multiple-fault case. The system could contain large uncertainties when the failure dynamics in (45) are large. Under this situation, the controller has to take an appropriate control action for the uncertainties occurring at any time instant \( \tau_{\nu} \). This is an adaptive control problem, in which controller parameters are adjusted based on the estimated plant parameters. The method based on the modified STC scheme is proposed to accomplish the fault tolerance control (FTC).

There are three assumptions of the proposed method that are addressed as follows
**Assumption 1**: The controlled system is controllable and observable even if faults occur.

**Assumption 2**: The control input is persistently excited.

**Assumption 3**: Before the fault occurs, the system is healthy or has fully recovered from the previous fault.

### 5.2 Modified active fault tolerance

The STC scheme should be modified to cope with parameter variations due to system faults. When a partial fault occurs, the system parameters vary accordingly. The estimated time-varying parameters obtained via the RELS algorithm in the conventional STC scheme would give large parameter errors and result in a poor system performance. However, based on the Kalman filter interpretation algorithm of RELS method [24], a modified scheme is proposed to estimate parameter variations. The above modified state-space self-tuning control scheme can be applied to the multivariable stochastic faulty system without prior message of system parameters and noise properties.

In short, in the beginning, a healthy and unknown system is well tuned by the modified STC scheme, and then the self-tuning structure with the reset covariance matrices of parameter estimate is modified to enhance the parameter estimation and tracking performance, when the system and/or inputs are partially faulty by [11].

It postulates that the estimated parameter is not constant but varies like a random walk

$$\theta_i(k) = \theta_i(k-1) + \bar{w}_i(k), \quad (46)$$

$$\epsilon_i(k) = y_i(k) - \hat{\theta}_i^T(k-1)\phi_i(k), \quad (47)$$

$$E\left[\bar{w}_i(k)\bar{w}_i^T(k)\right] = R_{ii}, \quad (48)$$

$$E\left[\epsilon_i^T(k)\epsilon_i(k)\right] = R_{2i}, \quad (49)$$

where $i = 1, 2, \cdots, p$, $\bar{w}_i(k)$ is the white Gaussian noise sequence. The Kalman filter then still gives the conditional expectation and covariance of $\theta_i(k)$ as

$$\hat{\theta}_i(k) = \hat{\theta}_i(k-1) + M_i(k)\epsilon_i^T(k), \quad (50)$$

$$M_i(k) = \frac{\bar{S}_i(k-1)\phi_i(k)}{R_{2i} + \phi_i^T(k)\bar{S}_i(k-1)\phi_i(k)}, \quad (51)$$

$$\bar{S}_i(k) = \bar{S}_i(k-1) - \frac{\bar{S}_i(k-1)\phi_i(k)\phi_i^T(k)\bar{S}_i(k-1)}{R_{2i} + \phi_i^T(k)\bar{S}_i(k-1)\phi_i(k)} + R_{ii}, \quad (52)$$

with

$$\bar{S}_i(0) = E\left[\hat{\theta}_i(0) - \theta_i(0)\right]\left[\hat{\theta}_i(0) - \theta_i(0)\right]^T. \quad (53)$$
\( \Sigma_i(k) \) is the covariance matrix of the parameter estimate \( \hat{\theta}_i(k) \). Usually, the estimated residual or the innovation error vector \( e_i(k) = y_i(k) - \hat{\theta}_i^T(k-1)\phi_i(k) \) will be near white if the model with parameter estimate is in good agreement with its true system. Here, we would like to point out that \( \hat{\theta}(k) \) contains the prediction control input \( u_{d,i}(k) \) as mentioned before.

To modify Kalman filter interpretation of RELS method, some appropriate initializations of \( R_{ii}, R_{2i}, \) and \( \Sigma_i(0) \) in (51)-(53) are assumed to be pre-specified before the parameter estimation process. When unanticipated system faults bring the process with large parameter variations, they need to be reasonably reset. To modify the conventional STC process with the RELS estimate algorithm for the faulty system, we propose to approximate \( R_{ii}, R_{2i}, \) and \( \Sigma_i(0) \) by the following moving window-based statistical quantities:

\[
R_{2i} \approx \frac{1}{N_h - k_i + 1} \sum_{k=k_i}^{N} e_i^T(k)e_i(k),
\]

and

\[
\Sigma_i(0) \approx \text{diag} \left\{ \frac{1}{N_h - k_i + 1} \sum_{k=k_i}^{N} \left[ \hat{\theta}_i(k) - \theta_i(N_h) \right] \left[ \hat{\theta}_i(k) - \theta_i(N_h) \right]^T \right\},
\]

where \( k_i \) is the time index after the estimate \( \hat{\theta}_i(k) \) in steady-state, and \( N_h \) is an appropriate time index for the healthy STC. It should be noted that the elements of \( \hat{\theta}_i(k) \) in (55) would not be independent with each other, when \( N_h - k_i + 1 \) in (55) is not large enough. Similarly, \( R_{ii} \) can be approximated by comparing (46) with (50) as follows

\[
R_{ii} \approx \left[ M_i(N_h) e_i^T(N_h) \right] \left[ M_i(N_h) e_i^T(N_h) \right]^T,
\]

where

\[
M_i(N_h) = \frac{\Sigma_i(0)\phi_i(N_h)}{R_{2i} + \phi_i^T(N_h)\Sigma_i(0)\phi_i(N_h)}.
\]

The proposed STC with the algorithm (50)-(53) and the initialization (54)-(56) works only for the plant with slowly time-varying parameters. This can be interpreted by the fact that the initialized \( R_{ii} \) obtained from (56) is so small while the system is healthy in general; hence \( \theta_i(k) \equiv \theta_i(k-1) \) in (46). As a result, it cannot reflect the real parameter variation induced by the unanticipated system faults. Therefore, \( \Sigma_i(k), R_{2i}, \) and \( R_{ii} \) in (52) need to be appropriately reset when a fault is detected at time instant \( k_f \). Although the algorithm with an appropriately reset forgetting factor \( \lambda(k_f) \) could improve estimations of
parameter variation for the conventional STC scheme, the reset forgetting factor $\lambda(k_f)$ would need some trials for various failure modes. Nevertheless, the resets of $S_i(k_f), R_{ii},$ and $R_{2i}$ proposed in [11] is a systematic approach for various failure modes.

Because the fact that the parameter variations induced by faults are unknown, the rule of thumb to reset the covariance matrices of the parameter estimate $\bar{S}_i(k)$ in (52) online is given as follows. When the fault be detected at time instant $k_f$, the variation of parameter estimations before and after the fault can be approximated as

$$\delta \hat{\Theta}(k_f) \approx \hat{\Theta}(k_f) - \hat{\Theta}(k_h), \quad \text{for} \quad k_h < k_f,$$

(57)

where $\hat{\Theta}(k_h)$ is the parameter estimate of the healthy system. Then, based on the physical interpretation of (53), $\bar{S}_i(k-1)$ in (52) can be reasonably reset as

$$\bar{S}_i(k_f-1) \equiv \text{diag} \left\{ \left[ \hat{\Theta}(k_f) - \hat{\Theta}(k_h) \right] \left[ \hat{\Theta}(k_f) - \hat{\Theta}(k_h) \right]^T \right\}$$

$$\approx \delta^2 \text{diag} \left[ \hat{\Theta}(k_f) \hat{\Theta}(k_f)^T \right].$$

(58)

Due to the additive uncertainties considered, we can assume the average parameter variation is in the range of the same order of magnitude of the fault system. So, it is reasonable to set $\delta = 1$, which denotes the worst case of this assumption. Some numerical examples are also given in [11] to show the sensitivity of various $\delta$'s to the mean value of the tracking error to verify the effectiveness of this rule of thumb for resetting the covariance matrices of the parameter estimate.

To improve the parameter estimation for the unanticipated faulty systems, $R_{2i}$ in (54) needs to be reset by a moving window with the residual

$$R_{2i} \approx \frac{1}{k_f - k_i + 1} \sum_{k=k_i}^{k_f} \epsilon_i(k) \epsilon_i(k),$$

(59)

where $1 < (k_f - k_i) < 5$ usually. Similarly, $R_{ii}$ in (53) should be reset by substituting (58) and (59) into (51) and then (50) to obtain

$$R_{ii} \approx \left[ M_i(k_f) \epsilon_i^T(k_f) \right] \left[ M_i(k_f) \epsilon_i^T(k_f) \right]^T.$$

(60)

In the STC scheme, the estimated residual is updated for every sampling time. Consequently, it is convenient to use it as the information of fault detection. Therefore, the time instant $k_f$ of the fault occurrence could be detected by utilizing the ratio with the residual $R_{2i}$ in (59) and the average norm of the innovation vectors as
\[ \frac{R_{2i}}{R_{fi}} > \gamma_{\epsilon i}, \]  
where \( R_{fi} = \frac{1}{k_f} \sum_{k=1}^{k_f} \varepsilon_i^T(k) \varepsilon_i(k) \) and \( \gamma_{\epsilon i} \) is a preset threshold.

The following statements summarize the FTC using the modified STC methodology with the fault detection and covariance matrices resetting:

1) Apply the modified ARMAX model-based STC algorithm to the healthy system until it well tracks the pre-specified trajectory. (for example, at time instant \( t = NT \), mentioned in (54) and (55).)

2) Switch the modified RELS estimation algorithm to the modified Kalman filter estimation algorithm (50)-(52) with initialized \( R_{li} \), \( R_{2i} \), and \( S_i(0) \) via (54)-(56).

3) Perform the modified STC scheme and the fault detection.

4) Whenever a fault is detected and the error is large enough for the ratio \( \frac{R_{2i}}{R_{fi}} > \gamma_{\epsilon i} \), reset \( R_{li} \), \( R_{2i} \), and \( S_i(k_f) \) by (58)-(60).

5) When the fault has recovered, go to Step 3) and repeat the modified STC process.

### 6. An illustration example

Consider a continuous-time linear singular system described as follows:

\[ \begin{align*}
E_{\mathcal{c}} \dot{x}_{\mathcal{c}}(t) &= A_{\mathcal{c}} x_{\mathcal{c}}(t) + B_{\mathcal{c}} u_{\mathcal{c}}(t), \\
y_{\mathcal{c}}(t) &= C_{\mathcal{c}} x_{\mathcal{c}}(t),
\end{align*} \]

where \( E_{\mathcal{c}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{\mathcal{c}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0.5 & -0.25 & 0 & -1 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_{\mathcal{c}} = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]

\[ C_{\mathcal{c}} = \begin{bmatrix} 0.1 & 0.5 & 5 & 0.01 & 0 & -0.001 \\
0 & 0 & 0 & 0 & 0.001 & 2.9995 \end{bmatrix}. \]

\[ x_{\mathcal{c}}(0) = M_{\mathcal{c}}(0) = M_{\mathcal{c}} \begin{bmatrix} \bar{x}_{\mathcal{c}}(0) \\
\bar{x}_{\mathcal{c}}(0) \end{bmatrix}^T = \begin{bmatrix} 0.4 & 0.2 & 0.5 & 0.25 & 0.5 \end{bmatrix}^T, \]

\[ \bar{x}_{\mathcal{c}}(0) = \begin{bmatrix} 0.4 & 0.2 & 0.5 \end{bmatrix}^T \text{ and } \bar{x}_{\mathcal{c}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T. \]

The singular system can be proved as a controllable and observable system by Mertzios et al. [3]. Since at the initial time \( t = 0 \), before the design process, \( u_{\mathcal{c}}(0) = 0_{2\times 1} \), it implies \( \bar{x}_{\mathcal{c}}(0) = -B_f u_{\mathcal{c}}(0) = 0_{2\times 1} \). However, at the initial time instant \( t = 0 \), one will design a non-zero control input \( u_{\mathcal{c}}(0) \) so that it will yield
the desired tracking purpose at \( t = 0^+ \). Nevertheless, this designed non-zero input \( u_c(0) \) can not be used to determine the initial state \( \vec{x}_0(0) = 0_{2x1} \). So, the initial condition should be given cautiously. The reference input is given by 
\[
r(t) = [\begin{array}{c}
-\sin(3\pi t) \\
1 - \cos(3\pi t)
\end{array}]^T.
\]

Multiplying \((\alpha E_r + \beta A_r)^{-1}\) to \( E_r \), \( A_r \), and \( B_r \), where \( \alpha = 2 \) and \( \beta = -1 \) for this case. Then, one can have the corresponding standard pencil as follows
\[
E_n \dot{x}_c(t) = A_n x_c(t) + B_n u_c(t),
\]
where \( E_n = (\alpha E_r + \beta A_r)^{-1} E_r \), \( A_n = (\alpha E_r + \beta A_r)^{-1} A_r \), and \( B_n = (\alpha E_r + \beta A_r)^{-1} B_r \).

Because \( E_n \) is singular, i.e., \( E_n \) includes some zero eigenvalues, utilizing the bilinear transform to find the similarity transformation matrix \( M \) of \( E_n \) is necessary. Assume \( \rho = 0.1 \), utilizing the algorithm described in the [25], one has
\[
\tilde{E}_n = (E_n - \rho I_n)(E_n + \rho I_n)^{-1}.
\]

Now, apply the extended matrix sign function to construct the block model transform matrix \( M \), which is described as follows:

\[
\begin{align*}
sign^+(\tilde{E}_n) &= \frac{1}{2} [I_n + \text{sign}(\tilde{E}_n)], \\
sign^-(\tilde{E}_n) &= \frac{1}{2} [I_n - \text{sign}(\tilde{E}_n)], \\
\text{sign}(\tilde{E}_n) &= \tilde{E}_n (\sqrt{\tilde{E}_n^2}^{-1}) = \tilde{E}_n^{-1}(\sqrt{\tilde{E}_n^2}), \\
M &= [\text{ind} \left( \text{sign}^+(\tilde{E}_n) \right) \; \text{ind} \left( \text{sign}^-(\tilde{E}_n) \right)] = \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}, \\
\tilde{E}_n \tilde{x}(t) &= \tilde{A}_n \tilde{x}(t) + \tilde{B}_n u(t),
\end{align*}
\]
where \( \tilde{E}_n = M^{-1} E_n M \), \( \tilde{A}_n = M^{-1} A_n M \), and \( \tilde{B}_n = M^{-1} B_n \).

Let
\[
\tilde{E}_n = \begin{bmatrix}
E_1 & 0_{4x2} \\
- & + \\
0_{2x4} & \tilde{E}_2
\end{bmatrix}.
\]
One can simplify the above equation by multiplying \( \tilde{E} = \text{block diag}\left([E_1^{-1} \\ I]\right) \).
on the left of the above equality to have

\[ E x(t) = A x(t) + B u(t), \]

in which \( E = \begin{bmatrix} E_1 & 0_{2c} \\ 0_{2d} & I_{2c} \end{bmatrix}, E = EE = \begin{bmatrix} l_{2d} & 0_{2c} \\ 0_{2d} & E_f \end{bmatrix}, A_s = EA_s, \) and \( B_s = EB = \begin{bmatrix} B_s \\ B_f \end{bmatrix}. \)

As a result, one has

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_c(t) \\
\dot{x}_f(t)
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & -0.5 & 0 & 0 \\
0 & 0.5 & -0.25 & -0.5 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_c(t) \\
x_f(t)
\end{bmatrix}
+ \begin{bmatrix}
0.5 & 0.5 \\
-0.25 & -0.25 \\
0.5 & 0.5 \\
0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
\dot{x}_c(t) \\
\dot{x}_f(t)
\end{bmatrix}
+ \begin{bmatrix}
u_c(t)
\end{bmatrix},
\]

where \( \dot{x}_c(t) = \begin{bmatrix} \dot{x}_{cs}(t) \\ \dot{x}_f(t) \end{bmatrix}. \)

Then, one has the slow subsystem \( \dot{x}_{cs}(t) = A_s x_{cs}(t) + B_s u_c(t) \) and the nondynamic fast system \( \dot{x}_f = -\beta B_j u_c(t) = B_j u_c(t), \)
in which \( A_s = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\
0 & 0.5 & -0.25 & -0.5 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0.5 \end{bmatrix}, B_s = \begin{bmatrix} 0.5 & 0.5 \\
-0.25 & -0.25 \\
0.5 & 0.5 \\
0.5 & 0.5 \end{bmatrix}, \) and \( B_f = \begin{bmatrix} -1 \\ -1 \\ 0.5 \\ -0.5 \end{bmatrix}. \)

The corresponding output function can be rewritten as

\[ y_c(t) = C x_c(t) \]
\[ = C x_{cs}(t) + C x_f(t) + D u_c(t), \]

where \( C = \begin{bmatrix} 0.1 & 0.5 & 5 & 0.009 \\ 0 & 0 & 3 & 0.001 \end{bmatrix} \) and \( D = \begin{bmatrix} -0.01 & -0.01 \\ 0.0005 & -0.0005 \end{bmatrix}. \)

Finally, the singular system can be reformed as the regular model as follows

\[ \dot{x}_{cs}(t) = A_s x_{cs}(t) + B_s u_c(t), \quad (62.a) \]
in which \( x_c(t) \in \mathbb{R}^4 \), \( x_c(0) = [0.4 \ 0.2 \ 0.5 \ 0.3]^T \), \( y_c(t) \in \mathbb{R}^2 \), \( y_c(0) = [2.6445 \ 1.5]^T \), and the control input \( u_c(t) \in \mathbb{R}^2 \). However, since the given system model is unknown for the designer, based on the off-line OKID, it’s desired to determine the appropriate order of the proposed ARMAX model as 4 for this example. Then, take the appropriate order of the two-input two-output modified ARMAX model for the self-tuning control as \( \rho = 2 \) so that \( 4/p = \rho \) is integer, which implies \( G_o(k) \in \mathbb{R}^4 \).

### 6.1 System identification by RELS method (two-input-two-output)

Assume the unknown two-input-two-output system is given as the above equation (62), and an appropriate two-input-two-output modified ARMAX model for on-line RELS method has been determined by the off-line OKID. Then, identify the system through RELS method (29) based on the initial parameters \( \theta(0) \) of the modified ARMAX model obtained by OKID. Notice that this modified ARMAX model could approximate the general class of MIMO systems, where \( n/p \) is not integer not just only for this specific example.

Because system (64) is unknown, only the data of input and output at sampling instants are available to identify the system. The sampling period \( T \) is 10 ms. Comparison of the system identification of RELS method with the modified ARMAX model based on the intuitive initial parameter \( \theta(0) = [I_{2 \times 2} \ 0_{2 \times 2} \ I_{2 \times 2} \ 0_{2 \times 2} \ 0_{2 \times 2} \ 0_{2 \times 2}]^T \) and the OKID-obtained initial parameter is shown as follows, in which \( \lambda_0 = 0.9 \), \( \lambda(0) = 0.9 \), and \( S_{1}(0) = S_{2}(0) = 10 \times I_{14} \) as follows.

1. **The intuitive initial parameters** \( \theta(0) = [I_{2 \times 2} \ 0_{2 \times 4} \ I_{2 \times 2} \ 0_{2 \times 4} \ 0_{2 \times 2}]^T \)

Let the system (63) be excited by the zero mean white noise input \( u(t) = [u_1(t) \ u_2(t)]^T \) with the covariance \( \text{diag} \begin{bmatrix} \text{cov}(u_1) & \text{cov}(u_2) \end{bmatrix} = \text{diag} \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \), where the sampling period \( T \) is 10 ms. Generally the initial parameter \( \theta(k) \) of the modified ARMAX model to be identified by the RELS (29) is given by \( \theta(0) = [I_{2 \times 2} \ 0_{2 \times 2} \ I_{2 \times 2} \ 0_{2 \times 2} \ 0_{2 \times 2} \ 0_{2 \times 2}]^T \), which presents \( \tilde{y}(k) \equiv I_2 y(k-1) + O_2 y(k-2) + O_2 u^*(k) + I_2 u(k-1) + O_2 u(k-2) + O_2 e(k-1) + O_2 e(k-2) \) for \( k = 1 \). The results of identification are shown in Fig. 4 and Fig. 5.
The initial parameters \( \theta(0) \) obtained by OKID

The system and observer gains \( G_o, H_o, C_o, D_o, \) and \( K_o \) in (43) are obtained by OKID, to have the initial parameter \( \theta(0) = [G_{o1}, G_{o2}, H_{o1}, H_{o2}, D_{o1}, D_{o2}]^T \), which is close to the convergent value of \( \theta(k) \). Let \( \lambda_0 = 0.9, \lambda(0) = 0.9 \), and \( S_1(0) = S_2(0) = 10 \times I_{1d} \). Based on the above obtained initial parameter \( \theta(0) \), RELS algorithm (29) is applied to identify this system, and the result of identification is shown in Fig 6.
From Fig. 4 – Fig. 6, it’s obvious that proposed identification through RELS method with OKID yields a better performance on the system identification.

6.2 Active fault tolerance using modified ARMAX model-based state-space self-tuning control

In fact, the system model is unknown; we have only information about input and output data. The sampling period $T$ is selected as $5 \text{ms}$ for the off-line OKID and the on-line state-space self-tuning control. First, determine an appropriate two-input-two-output modified ARMAX model via the off-line OKID for the self-tuning control with fault tolerance using the modified estimation algorithm and the fault detection is used to adapt to the fault tolerance control. Notice that the initial parameter $\theta(0)$ of the determined ARMAX model for the on-line RELS is obtained by OKID. If a fault is detected at $t = k_f$, matrices $R_{1i}, R_{2i}$, and $\bar{S}_i(k_f)$ with the reset parameter $\delta = 1$ in (58) is automatically reset again. The fault detection thresholds are $\gamma_{e1} = 3.5$ and $\gamma_{e2} = 3.5$ in (61). The proposed two-input-two-output modified ARMAX model is then given by

$$
\begin{align*}
\begin{bmatrix} \hat{y}_1(k) \\ \hat{y}_2(k) \end{bmatrix} &= 
\begin{bmatrix} g_{111} & g_{112} \\ g_{121} & g_{122} \end{bmatrix} y_{d1}(k-1) + 
\begin{bmatrix} g_{211} & g_{212} \\ g_{221} & g_{222} \end{bmatrix} y_{d2}(k-1) \\
&+ 
\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} u_{d1}(k) + 
\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} u_{d2}(k) \\
&+ 
\begin{bmatrix} d_{111} & d_{112} \\ d_{121} & d_{122} \end{bmatrix} \varepsilon_1(k-1) + 
\begin{bmatrix} d_{211} & d_{212} \\ d_{221} & d_{222} \end{bmatrix} \varepsilon_2(k-1)
\end{align*}
$$
Active low-order fault-tolerant state space self-tuner

\[
= G_{o1} \begin{bmatrix} y_{d1}(k-1) \\ y_{d2}(k-1) \end{bmatrix} + G_{o1} \begin{bmatrix} y_{d1}(k-2) \\ y_{d2}(k-2) \end{bmatrix} + H_{o0} \begin{bmatrix} u_{d1}^*(k) \\ u_{d2}^*(k) \end{bmatrix} \\
+ H_{o1} \begin{bmatrix} u_{d1}(k-1) \\ u_{d2}(k-1) \end{bmatrix} + H_{o2} \begin{bmatrix} u_{d1}(k-2) \\ u_{d2}(k-2) \end{bmatrix} + D_{o1} \begin{bmatrix} e_1(k-1) \\ e_2(k-1) \end{bmatrix} + D_{o2} \begin{bmatrix} e_1(k-2) \\ e_2(k-2) \end{bmatrix}, (63.a)
\]

where \( u_d^*(k) = \begin{bmatrix} u_{d1}^*(k) \\ u_{d2}^*(k) \end{bmatrix} = -K_d(k-1)x_o(k) + E_d(k-1)r^*(k) \) is the prediction of the control input, \( x_o(k) = G_o(k-1)x_o(k-1) + H_o(k-1)u_d(k-1) + K_o(k-1)e_o(k-1) \), \( x_o(0) = C_o^\top y_o(0) = [0.1 \quad 0.1]^\top \), and \( r^*(k) = r(k+1) \).

Simplify the model to form a linear-in-the-parameters expression

\[
\tilde{y}_1(k) = \theta_1^T(k)\phi_1(k), \\
\tilde{y}_2(k) = \theta_2^T(k)\phi_2(k), \quad (63.b)
\]

where

\[
\theta_1(k) = \begin{bmatrix} g_{111} & g_{112} & \ldots & g_{121} & h_{111} & h_{112} & \ldots & h_{211} & d_{111} & d_{112} & \ldots & d_{212} \end{bmatrix}^T \in \mathbb{R}^{d_1d}, \\
\theta_2(k) = \begin{bmatrix} g_{211} & g_{212} & \ldots & g_{221} & h_{211} & h_{212} & \ldots & h_{222} & d_{211} & d_{212} & \ldots & d_{222} \end{bmatrix}^T \in \mathbb{R}^{d_2d}, \\
\theta(k) = \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \end{bmatrix},
\]

\( \phi_1(k) \) and \( \phi_2(k) \) are the related terms to parameters \( \theta_1(k) \) and \( \theta_2(k) \), respectively. Estimate the parameters \( \theta_1(k) \) and \( \theta_2(k) \) using the standard recursive extended-least-squares (RELS) algorithm. The initial parameter \( \theta(0) \) is obtained by OKID, where \( \lambda_0 = 0.9 \), \( \lambda(0) = 0.9 \), \( S_1(0) = S_2(0) = 10 \times I_{14} \) and the weighting matrix of cost function \( W = 5.5I_2 \) determined by the off-line preprocess of the on-line self-tuning control, where \( r(t) = [\sin(t) \quad 1 - \cos(t)]^\top \).

Matrices \( G_o(k) \) and \( H_o(k) \) in the associated state-space innovation form (45) are determined as

\[
G_o(k) = \begin{bmatrix} -G_{o1}(k) & I_2 \\ -G_{o2}(k) & 0_2 \end{bmatrix}, H_o(k) = \begin{bmatrix} H_{o1}(k) \\ H_{o2}(k) \end{bmatrix}, C_o = [I_2 \quad 0_2], \text{ and } D_o = H_{o0}(k), \quad (64)
\]

where

\[
G_{o1} = \begin{bmatrix} g_{111} & g_{112} \\ g_{211} & g_{212} \end{bmatrix}, G_{o2} = \begin{bmatrix} g_{311} & g_{312} \\ g_{411} & g_{412} \end{bmatrix}, H_{a1}(k) = \begin{bmatrix} h_{111} & h_{112} \\ h_{211} & h_{212} \end{bmatrix}, H_{a2}(k) = \begin{bmatrix} h_{311} & h_{312} \\ h_{411} & h_{412} \end{bmatrix}, H_{o1}(k) = \begin{bmatrix} h_{111} & h_{112} \\ h_{211} & h_{212} \end{bmatrix}, H_{o2}(k) = \begin{bmatrix} h_{311} & h_{312} \\ h_{411} & h_{412} \end{bmatrix}
\]

and weighting matrices \( \{Q = 10^5 \times I_2, R = I_2\} \) in (5) and \( \{Q_o = 10^5 \times I_4, R_o = I_2\} \) in (40). The FTC with an abrupt input fault and a gradual input fault is considered.
6.2.1 Determination of the weighting matrix

![Figure 7](image)

**Fig. 7** The convergence of error vs. weighting matrices.

One can see that the weighting factor with respect to the minimum error is 5.5, so we choose \( W = wI_2 = 5.5I_2 \) for this example.

6.2.2 Modified ARMAX model-based state-space self-tuning control

Output responses based on the novel and the traditional discrete linear quadratic tracker for self-tuning control is respectively given as follows. It shows the proposed one has a quite well tracking performance; however, the traditional one yields a divergent output response.

![Figure 8](image)

**Fig. 8** Output responses based on (a) the novel discrete linear quadratic tracker vs. (b) the traditional discrete linear quadratic tracker.

6.2.3 Fault scenario 1, an abrupt input fault

In the beginning, we use the modified STC methodology to control the linear singular system, and then switch it to the FTC at the 0.5th second. At the 2.5th second, the Input 1 and Input 2 are assumed to be abruptly added to 0.55 times
and 0.5 times of their functions, respectively, then simulation results are given in Fig. 9. And the simulation results without using the FTC are shown in Fig. 10.

![Fig 9](image1.png) **Fig 9** The output responses with an abrupt input fault using the FTC with the modified STC methodology.

![Fig 10](image2.png) **Fig 10** The output responses with an abrupt input fault without any FTC.

### 6.2.4 Fault scenario 2, a gradual input fault

In the beginning, we use the modified STC methodology to control the linear singular system, and then switch it to the FTC at the 0.5th second. At the 2.5th second, the Input 1 and Input 2 are assumed to be abruptly added to 0.5\((1+(1-e^{-\left(t-2.5\right)}))\) times and 0.5\((1+(1-e^{-\left(t-2.5\right)}))\) times of their functions, respectively, then simulation results are given in **Fig. 11**. And the simulation results without using the FTC are shown in **Fig. 12**.

![Fig 11](image3.png) **Fig. 11** The output responses with a gradual input fault using the FTC with the modified STC methodology.
Fig. 12 The output responses with a gradual input fault without any FTC.

7 Conclusion

A novel fault-tolerant state-space self-tuner for the unknown sample-data linear regular system with a direct feedthrough term has been proposed in this paper. The main contributions of this paper are i) A novel digital tracker for the equivalent linear singular system without impulse mode is proposed, ii) Embedded with the prediction based control input for pure system identification only, the modified ARMAX model-based system identification is presented for the unknown linear singular systems, iii) Initialization for on-line system identification obtained by the off-line OKID is proposed to speed up the parameter identification process, iv) A modified state-space innovation form is presented to integrate with above advantages to yield a modified ARMAX model-based state space self-tuner for the unknown linear singular system, and v) The above state-space self-tuner has been further extended to the one with fault tolerance function, when the abrupt fault and/or the gradual input fault occur.

Appendix A. The principal $n$th root of a matrix and the associated matrix sector function [26, 27, 28].

Definition A.1
Let a matrix $A \in \mathbb{C}^{m \times m}$, eigenspectrum $\sigma(A) = \{\lambda_i, i = 1, 2, \ldots, m\}$, eigenvalues $\lambda_i \neq 0$ and $\arg(\lambda_i) \neq \pi$. The principal $n$th root of $A$ is defined as $\sqrt[n]{A} \in \mathbb{C}^{m \times m}$, where $n$ is a positive integer and (a) $\left(\sqrt[n]{A}\right)^n = A$, (b) $\arg(\sigma(\sqrt[n]{A})) \in (-\pi / n, \pi / n)$. The principal $n$th root of a matrix is unique.

Definition A.2
Let a matrix $A \in \mathbb{C}^{m \times m}$, $\sigma(A) = \{\lambda_i, i = 1, 2, \ldots, m\}$, $\lambda_i \neq 0$ and
arg(\(\lambda_i\)) \neq 2\pi(k + 1/2)/n \quad \text{for} \quad k \in [0,1]. \quad \text{In addition, let } \ M \quad \text{be a modal matrix of} \quad A, \quad \text{i.e.} \quad A = MJ_A M^{-1}, \quad \text{where} \quad J_A \quad \text{is a matrix containing Jordan blocks of} \quad A. \quad \text{Then, the matrix sector function of} \quad A, \quad \text{denoted by} \quad \text{sector}_n(A) \quad \text{or} \quad S_n(A), \quad \text{is defined as}

\[
\text{sector}_n(A) = S_n(A) = M \begin{bmatrix}
S_n(\lambda_1) & 0 & \cdots & 0 \\
0 & S_n(\lambda_2) & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & S_n(\lambda_m)
\end{bmatrix} M^{-1}, \quad (A1)
\]

where \(S_n(\lambda_i)\) is the scalar \(n\)-sector function of \(\lambda_i\), which is defined as
\(S_n(\lambda_i) = e^{i2\pi k/n} = \lambda_i \sqrt[n]{\lambda_i^n} \quad \text{with} \quad k \in [0,1] \quad \text{where} \quad \lambda_i \quad \text{lies in the interior of the minor sector in} \quad C \quad \text{bounded by the sector angles} \quad 2\pi(k-1/2)/n \quad \text{and} \quad 2\pi(k+1/2)/n, \quad \text{and} \quad \sqrt[n]{\lambda_i^n} \quad \text{is the principal} \quad n\text{th root of} \quad \lambda_i^n. \quad \text{When} \quad n = 2, \quad \text{the scalar} \quad n\text{-sector function of} \quad \lambda_i \quad \text{becomes the scalar sign function of} \quad \lambda_i, \quad \text{denoted by} \quad \text{Sign}(\lambda_i), \quad \text{i.e.} \quad \text{sector}_2(\lambda_i) = S_2(\lambda_i) = e^{i\pi} = \lambda_i \sqrt[n]{\lambda_i^n} = \text{sign}(\lambda_i) \quad \text{with} \quad k \in [0,1].

The matrix sector function \(S_n(A)\) defined in \textit{Definition A.2} can be expressed as
\[S_n(A) = A \left(\sqrt[n]{A^n}\right)^{-1}, \quad (A2)\]
where \(\sqrt[n]{A^n}\) is the principal \(n\)th root of \(A^n\). Also, the associated matrix sign function, denoted by \(\text{sign}(A)\), becomes
\[\text{sign}(A) = S_2(A) = A \left(\sqrt[n]{A^n}\right)^{-1}. \quad (A3)\]

The partitioned matrix sector function of \(A\) can be described as follows:

\textit{Definition A.3}

Let a matrix \(A \in C^{m \times m}\), \(\sigma(A) = \{\lambda_i; i = 1,2,\cdots,m\}\), \(\lambda_i \neq 0\) and \(\arg(\lambda_i) \neq 2\pi(p + 1/2)/n \quad \text{for} \quad p \in [0,n-1]. \quad \text{Also, let} \quad M \quad \text{be a modal matrix of} \quad A. \quad \text{Then, the} \quad q\text{th matrix} \quad n\text{-sector function of} \quad A, \quad \text{denoted by} \quad S_{n}(^{(q)}(A), \quad \text{is defined as}

\[
S_{n}(^{(q)}(A) = M \begin{bmatrix}
S_{n}(^{(q)}(\lambda_1) & 0 & \cdots & 0 \\
0 & S_{n}(^{(q)}(\lambda_2) & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & S_{n}(^{(q)}(\lambda_m)
\end{bmatrix} M^{-1}, \quad (A4)
\]

where \(q\text{ th scalar} \quad n\text{-sector function of} \quad \lambda_i, \quad \text{denoted by} \quad S_{n}(^{(q)}(\lambda_i), \quad \text{is}
\[ S_n^{(q)}(\lambda_i) = \begin{cases} 1,2\pi(q-1/2)/n < \arg(\lambda_i) < 2\pi(q+1/2)/n \text{ for } q \in [0,n-1], \\ 0, \text{ otherwise.} \end{cases} \]

The \( q \)th matrix \( n \)-sector function of \( A \) can be obtained by the following equation

\[ S_n^{(q)}(A) = \frac{1}{n} \sum_{i=1}^{n} [S_n(A)e^{-j2q\pi/n}]^{i-1} \text{ for } q \in [0,n-1]. \quad (A5) \]

When \( n = 2 \) and \( q = 1 \), the \( q \)th matrix \( n \)-sector function of \( A \) becomes the matrix sign minus function of \( A \), denoted by \( \text{sign}^{-}(A) \) or \( S_2^{(1)}(A) \) as

\[ \text{sign}^{-}(A) = S_2^{(1)}(A) = \frac{1}{2} [I_{m} - \text{sign}(A)]. \quad (A6a) \]

The matrix sign plus function of \( A \) is

\[ \text{sign}^{+}(A) = I_{m} - \text{sign}^{-}(A). \quad (A6b) \]

A generalized fast and stable algorithm with \( r \)-th-order convergence rate for computing the principal \( n \)th root of a given matrix \( A \) is listed below:

When the order of the desired convergence rate \( r = 2 \),

\[ G(l+1) = G(l) \left\{ \left[ 2I_{m} + (n-2)G(l) \right] \left[ I_{m} + (n-1)G(l) \right]^{-1} \right\}^{n}, \quad (A7a) \]

\[ G(0) = A, \quad \lim_{l \to \infty} G(l) = I_{m}; \]

\[ R(l+1) = R(l) \left[ 2I_{m} + (n-2)G(l) \right]^{-1} \left[ I_{m} + (n-1)G(l) \right], \quad (A7b) \]

\[ R(0) = I_{m}, \quad \lim_{l \to \infty} R(l) = \sqrt[n]{A}. \]

When the order of the desired convergence rate \( r = 3 \),

\[ G(l+1) = G(l) \left\{ \left[ 3I_{m} + \left( \frac{n^2+5n-12}{2} \right) G(l) + \left( \frac{n^2-5n+6}{2} \right) G^2(l) \right] \right\}^{n}, \quad (A8a) \]

\[ G(0) = A, \quad \lim_{l \to \infty} G(l) = I_{m}; \]

\[ R(l+1) = R(l) \left[ 3I_{m} + \left( \frac{n^2+5n-12}{2} \right) G(l) + \left( \frac{n^2-5n+6}{2} \right) G^2(l) \right]^{-1} \]

\[ \times \left[ I_{m} + \left( \frac{n^2+3n-4}{2} \right) G(l) + \left( \frac{n^2-3n+2}{2} \right) G^2(l) \right], \quad (A8b) \]

\[ R(0) = I_{m}, \quad \lim_{l \to \infty} R(l) = \sqrt[3]{A}. \]

Also, a generalized fast and stable algorithm with \( r \)-th-order convergence rate for computing the matrix sector function of a given matrix \( A \) is as follows:

When the order of the desired convergence rate \( r = 2 \),
Active low-order fault-tolerant state space self-tuner

\[ Q(l+1) = Q(l) \left[ 2I_m + (n-2)Q^*(l) \right] \left[ I_m + (n-1)Q^*(l) \right]^{-1}, \quad (A9) \]

\[ Q(0) = A, \quad \lim_{l \to \infty} Q(l) = S_\alpha(A). \]

When the order of the desired convergence rate \( r = 3 \)

\[ Q(l+1) = Q(l) \left[ 3I_m + \frac{n^2 + 5n - 12}{2} Q^*(l) + \frac{n^2 - 5n + 6}{2} Q^{2n}(l) \right] \]

\[ \times \left[ I_m + \frac{n^2 + 3n - 4}{2} Q^*(l) + \frac{n^2 - 3n + 2}{2} Q^{2n}(l) \right]^{-1}, \quad (A10) \]

\[ Q(0) = A, \quad \lim_{l \to \infty} Q(l) = S_\alpha(A). \]

Appendix B. Singular system descriptions

B.1 Preliminaries for decomposition of singular systems

Consider the singular system as follows

\[ E_r \dot{x}(t) = A_r x(t) + B_r u(t), \quad (B.1a) \]

\[ y(t) = C_r x(t), \quad (B.1b) \]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the input and \( y(t) \in \mathbb{R}^p \) is the output. These constant matrices \( E_r, A_r, B_r, \) and \( C_r \) all have appropriate dimensions, and \( E_r \) is a singular matrix. The matrix sign function of a square matrix \( A \in \mathbb{C}^{n \times n} \) with \( \text{Re}(\sigma(A)) \neq 0 \) is defined \([29]\) as follows

\[ \text{sign}(A) = 2\text{sign}^+(A) - I_n, \quad (B.2) \]

where \( I_n \) is an \( n \times n \) identity matrix and

\[ \text{sign}^+(A) = \frac{1}{2\pi i} \int_{C_+} (\lambda I_n - A)^{-1} d\lambda, \quad (B.3) \]

where \( C_+ \) is a simple closed contour in right-half plane of \( \lambda \) and encloses all right-half plane eigenvalues of \( A \). For another thing, the matrix sign function \([30, 31]\) is also defined as

\[ \text{sign}(A) = A(\sqrt{A^2})^{-1} = A^{-1}(\sqrt{A^2}), \quad (B.4) \]

where the matrix \( \sqrt{A^2} \) denotes the principal square root of \( A^2 \).

Preserving the eigenvectors of the original matrix is a main feature of the matrix sign function. This property is useful for engineering problem to study the eigenstructures of matrices and develop applications. The singular matrix \( E_r \) can be modified by using bilinear transformation.

\[ \tilde{E}_r = (E_r - \rho I_n)(E_r + \rho I_n)^{-1}, \quad (B.5) \]

where \( \rho \) is the radius of a circle with center at the origin so that the circle only contains zero eigenvalues and no eigenvalues of \( E_r \) located on the circle. Therefore, the eigenvalues within the circle are mapped into the left-half plane of
the complex $s$-plane, and the others are mapped into the right-half plane of the complex $s$-plane.

### B.2 The regular pencil and the standard pencil

**Definition B.2.1 [32]:**

Let $E_r$ and $A_r$ be two square constant matrices. If $\det(sE_r - A_r) \neq 0$, for all $s$, then $(sE_r - A_r)$ is called a regular pencil.

**Definition B.2.2 [33]:**

Let $(sE_n - A_n)$ be a regular pencil. If there exists scalars $\alpha$ and $\beta$ such that $\alpha E_n + \beta A_n = I_n$, then $(sE_n - A_n)$ is called a standard pencil. Note that for any regular pencil, $(sE_r - A_r)$ can be easily transformed into a standard one by multiplying $(\alpha E_r + \beta A_r)^{-1}$ to $E_r$ and $A_r$ respectively, where $\alpha$ and $\beta$ are scalars such that $\det(\alpha E_r + \beta A_r) \neq 0$. Hence, the matrix coefficients of a standard pencil $(sE_n - A_n)$ becomes

\[
E_n \triangleq (\alpha E_r + \beta A_r)^{-1} E_r, \quad (B.6)
\]
\[
A_n \triangleq (\alpha E_r + \beta A_r)^{-1} A_r. \quad (B.7)
\]

The modified system retains its state vector $x(t)$ and the matrices $(E_n, A_n)$ have the following properties.

**Lemma B.2.1 [34]:**

1): $E_n A_n = A_n E_n$, which means that $E_n$ and $A_n$ commute each other.

2): $E_n$ and $A_n$ have the same eigenspaces.

The above properties enable us to decompose a singular system into a reduced-order regular subsystem (slow subsystem) and a nondynamic subsystem (fast subsystem).

### B.3 Decomposition of singular systems

Consider the continuous-time singular system (B.1). It is well known that the singular system can be decomposed into slow and fast subsystem. From (B.6) and (B.7), the regular pencil $(sE_r - A_r)$ can be transformed into a standard one $(sE_n - A_n)$. Note that since $E_r$ is a singular matrix, which has at least one zero eigenvalue, $\beta$ cannot be equal zero. Hence, multiply (B.1a) by $(\alpha E_r + \beta A_r)^{-1}$ can get the following equation

\[
E_n \dot{x}(t) = A_n x(t) + B_n u(t), \quad (B.8)
\]

where $E_n = (\alpha E_r + \beta A_r)^{-1} E_r$ and $A_n = (\alpha E_r + \beta A_r)^{-1} A_r$. Because $\alpha E_n + \beta A_n = I_n$, the pencil $(sE_n - A_n)$ is a standard one which has the properties mentioned in Lemma 1. In order to
decompose system (B.8), we convert state space $x(t)$ into $\bar{x}(t)$ by
\[ x(t) = M\bar{x}(t), \quad (B.9) \]
where the constant matrix $M$ is a block modal matrix of $E_n$ and determined by means of the extended matrix sign function. The $M$ matrix of state space transformation is given as follows

**Step 1:** Find $\text{sign}(\tilde{E}_n)$ using the extended matrix sign function with an adequate $\rho$, where $\tilde{E}_n = (E_n - \rho I_n)(E_n + \rho I_n)^{-1}$.

**Step 2:** Find $\text{sign}^+(\tilde{E}_n) = \frac{1}{2}[I_n + \text{sign}(\tilde{E}_n)]$ and $\text{sign}^-(\tilde{E}_n) = \frac{1}{2}[I_n - \text{sign}(\tilde{E}_n)]$.

**Step 3:** Construct the matrix $M = [\text{ind}(\text{sign}^+(\tilde{E}_n)) \text{ind}(\text{sign}^-(\tilde{E}_n))]$, where $\text{ind}(\cdot)$ represents the collection of the linearly independent column vectors of $\cdot$.

Substituting (B.9) into (B.8) and multiplying $M^{-1}$ on the left, the equation can be rewritten as
\[ M^{-1}E_nM\bar{x}(t) = M^{-1}A_nM\bar{x}(t) + M^{-1}B_nu(t) \]
\[ = M^{-1}\frac{1}{\beta}(I_n - \alpha E_n)M\bar{x}(t) + M^{-1}B_nu(t) \]
i.e.
\[ \begin{bmatrix} \bar{E}_1 & O \\ O & \bar{E}_2 \end{bmatrix} \bar{x}(t) = \begin{bmatrix} \frac{1}{\beta}(I_k - \alpha \bar{E}_1) & O \\ O & \frac{1}{\beta}(I_{n-k} - \alpha \bar{E}_2) \end{bmatrix} \bar{x}(t) + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u(t), \quad (B.10) \]
where $\bar{x}(t) = [\bar{x}_1^T(t), \bar{x}_2^T(t)]^T$, and $M^{-1}E_nM=$block diagonal $\{\bar{E}_1, \bar{E}_2\}$. $\bar{E}_1$ is invertible with $\text{rank}(\bar{E}_1) = \text{deg}(|\det(s\bar{E}_1 - A_1)|) \equiv \kappa$, $[\bar{B}_1^T, \bar{B}_2^T]^T = M^{-1}B_n$ and $\bar{E}_2$ is a nilpotent matrix with dimension $(n-\kappa) \times (n-\kappa)$. Notice that since $\det(I_{n-k} - \alpha \bar{E}_2) = 1$, it is invertible. Simplifying (B.10) by pre-multiplying the block diagonal $\left\{ \bar{E}_1^{-1}, \beta(I_{n-k} - \alpha \bar{E}_2)^{-1} \right\}$ on both sides, one has
\[ \begin{bmatrix} I_k & O \\ \beta(I_{n-k} - \alpha \bar{E}_2)^{-1} & \bar{E}_2 \end{bmatrix} \bar{x}(t) = \begin{bmatrix} \frac{1}{\beta}(\bar{E}_1^{-1} - \alpha I_k) & O \\ O & I_{n-k} \end{bmatrix} \bar{x}(t) + \begin{bmatrix} \bar{E}_1^{-1}\bar{B}_1 \\ \beta(I_{n-k} - \alpha \bar{E}_2)^{-1}\bar{B}_2 \end{bmatrix} u(t), \]
\[ \begin{bmatrix} I_k & O \\ \beta(I_{n-k} - \alpha \bar{E}_2)^{-1} & \bar{E}_2 \end{bmatrix} \tilde{x}(t) = \begin{bmatrix} A_s & O \\ O & I_{n-k} \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} \bar{B}_s \\ \bar{B}_f \end{bmatrix} u(t), \quad (B.11) \]
where $\bar{x}(t) = \beta(I_{n-k} - \alpha \bar{E}_2)^{-1}\tilde{x}(t)$, $\bar{x}(t) = \beta(I_{n-k} - \alpha \bar{E}_2)^{-1}\tilde{x}(t)$, $\bar{x}(t) = \beta(I_{n-k} - \alpha \bar{E}_2)^{-1}\tilde{x}(t)$, and $\bar{x}(t) = \beta(I_{n-k} - \alpha \bar{E}_2)^{-1}\tilde{x}(t)$. To transfer the singular system (B.11) into an equal
regular model, we discuss two cases.

**Case 1:** $\bar{E}_f$ is a zero matrix:

System (B.11) can be written as

$$\begin{bmatrix} I_{\kappa} & O \\ O & \bar{O} \end{bmatrix} \dot{x}(t) = \begin{bmatrix} A_{\kappa} & O \\ O & I_{n-\kappa} \end{bmatrix} \bar{x}(t) + \begin{bmatrix} \bar{B} \\ \bar{B}_f \end{bmatrix} u(t),$$  

(B.12)

and (B.1b) can be rewritten as

$$y(t) = C_r x(t) = C_r M \bar{x}(t) \hat{\mathbf{x}}_C \bar{x}(t),$$  

(B.13)

where $C = C_r M$ and $\bar{x}(t) = \begin{bmatrix} \bar{x}_r(t) \\ \bar{x}_f(t) \end{bmatrix}$. Then, the continuous model (B.12) and (B.13) can be converted as follows

$$\dot{\bar{x}}_r(t) = \bar{A}_{\kappa} \bar{x}_r(t) + \bar{B}_r u(t),$$  

(B.14)

$$\bar{x}_f(t) = -\bar{B}_f u(t),$$  

(B.15)

$$y(t) = C \bar{x}(t) = [C_1 \quad C_2] \begin{bmatrix} \bar{x}_r(t) \\ \bar{x}_f(t) \end{bmatrix} \tag{B.16}$$

where $\bar{C} = C_1 \in R^{p \times k}$, $C_2 \in R^{p \times (n-k)}$, and $D = -C_2 \bar{B}_f \in R^{p \times m}$.

**Case 2:** $\bar{E}_f$ is a Jordan matrix.

It is remarkable to note that since

$$\text{rank}(E_r) - \text{deg}\{\text{det}(\bar{S}E_r - A_r)\} = \text{rank}(\bar{E}_f),$$  

(B.17)

it is much easier to determine the number of the impulse modes for using the above equation relating to (B.11). First at all, we assume the singular system has $q$ impulsive modes, and then $\text{rank}(\bar{E}_f) = q$. What follows is to find preliminary feedback gain $K_{\text{inner}}$ and to prove $K_{\text{inner}}$ can eliminate impulsive modes by following steps.

Let

$$\bar{x}(t) = V \hat{x}(t),$$  

(B.18)

where $\hat{x}(t) = \begin{bmatrix} \hat{x}_r^T(t), \hat{x}_f^T(t) \end{bmatrix}^T = \begin{bmatrix} \bar{x}_r^T(t), (U^{-1} \bar{x}_f(t))^T \end{bmatrix}^T$ and $V = \begin{bmatrix} I_k & O \\ O & \bar{U} \end{bmatrix}$. $U$ is a modal matrix of $E_f$ with dimension $(n-\kappa) \times (n-\kappa)$ such that $U^{-1} \bar{E}_f U$ is in the Jordan block form. Substituting (B.18) into (B.11) and pre-multiplying it by $V^{-1}$, we obtain the following equation.
\[
\begin{bmatrix}
I_k & \frac{1}{\hat{E}_f} \hat{O} \\
\hat{O} & \frac{1}{\hat{E}_f}
\end{bmatrix}
\tilde{x}(t) = \begin{bmatrix}
\hat{A}_f & \frac{1}{\hat{E}_f} \\
\frac{1}{\hat{E}_f} & \frac{1}{\hat{E}_f}
\end{bmatrix} \tilde{x}(t) + \begin{bmatrix}
\hat{B}_f \\
\frac{1}{\hat{E}_f}
\end{bmatrix} u(t),
\quad (B.19)
\]

where \( \hat{E}_f = U^{-1} \overline{E}_f U \), \( \hat{A}_f = \overline{A}_f \), \( \hat{B}_f = \overline{B}_f \), and \( \hat{B}_f = U^{-1} \overline{B}_f \). Notice that \( \hat{E}_f \) is in the Jordan block form with \( d \) blocks of sizes \( u_1, u_2, \ldots, u_d \), where \( \sum_{i=1}^{d} u_i \) is the column(row) number of \( \hat{E}_f \). Taking the Laplace transformation of the fast subsystem \( \hat{E}_f \hat{x}_f(t) = \hat{x}_f(t) + \hat{B}_f u(t) \) in (A.19), one obtains
\[
\hat{x}_f(s) = (s \hat{E}_f - I_{n-\kappa})^{-1}(\hat{E}_f \hat{x}_f(0) + \hat{B}_f U(s))
\]
\[
= -\sum_{i=0}^{l-1} s^i \hat{E}_f^i (\hat{E}_f \hat{x}_f(0) + \hat{B}_f U(s)),
\quad (B.20)
\]
where \( \hat{x}_f(s) \) and \( U(s) \) denote the Laplace transformations of \( \hat{x}_f(t) \) and \( u(t) \) respectively, \( \hat{x}_f(0) \) denotes the initial value of \( \hat{x}_f(t) \), and \( l \) is said to be the nilpotency index of \( \hat{E}_f \). Taking the inverse Laplace transformation of above equation, one has
\[
\hat{x}_f(t) = -\sum_{i=0}^{l-1} \hat{E}_f^i \hat{x}_f(0) \delta^{(i-1)}(t) - \sum_{i=0}^{l-1} \hat{E}_f^i \hat{B}_f u^{(i)}(t),
\quad (B.21)
\]
where \( \delta(t) \) and \( \delta^{(i)}(t) \) denote the delta function and the \( i \)th derivative of the delta function respectively. From the above equation (B.21), the impulsive modes of the fast state result from inconsistent initial conditions of the fast state or discontinuous control input (or its derivatives) can be obviously observed.

Here, we propose a preliminary feedback design method to eliminate the impulsive modes. Determining the preliminary feedback gain \([25] K_{\text{inner}} = [K_{\text{inner}1}, K_{\text{inner}2}, \ldots, K_{\text{inner}n-\kappa}]_{m \times (n-\kappa)} \), where \( K_{\text{inner}j} \) is of dimension \( m \times 1 \) for \( j = 1, 2, \ldots, (n-\kappa) \), is summarized as follows.

1. If \( u_i \geq 1 \), where \( 1 \leq i \leq d \), and its corresponding Jordan block is a null matrix, then
\[
K_{\text{inner}j+u_2+\cdots+u_i+1} = O_{m \times 1},
\]
\[
K_{\text{inner}j+u_2+\cdots+u_i+2} = O_{m \times 1},
\]
\[
\vdots
\]
\[
K_{\text{inner}j+u_2+\cdots+u_i+n-\kappa} = O_{m \times 1}.
\]
2. If \( u_i > 1 \), where \( 1 \leq i \leq d \) and its corresponding Jordan block is not a null matrix, then
\[
K_{\text{inner}}^{q+1} = \begin{bmatrix}
\delta(\hat{b}(u_1 + u_2 + \cdots + u_l)) \\
\delta(\hat{b}(u_1 + u_2 + \cdots + u_l)) \\
\vdots \\
\delta(\hat{b}(u_1 + u_2 + \cdots + u_l)) \\
\end{bmatrix},
\]

\[
K_{\text{inner}}^{q+2} = O_{m \times 1},
\]

\[
K_{\text{inner}}^{q+1} = O_{m \times 1},
\]

where

\[
\hat{B}_f = \begin{bmatrix}
\hat{b}_{x+1} \\
\hat{b}_{x+2} \\
\vdots \\
\hat{b}_n
\end{bmatrix},
\]

\[
\hat{b}_i = \begin{bmatrix}
\hat{b}_{i1} \\
\hat{b}_{i2} \\
\vdots \\
\hat{b}_{im}
\end{bmatrix},
\]

\[
\delta(\hat{b}_{ij}) = \begin{cases}
0 & \text{if } \hat{b}_{ij} = 0 \\
1 & \text{if } \hat{b}_{ij} > 0 \\
-1 & \text{if } \hat{b}_{ij} < 0
\end{cases}.
\]

Let

\[
u(t) = -K_{\text{inner}} \hat{\chi}(t) + v(t) \tag{B.22}
\]

Substituting (B.16) into (B.14) yields

\[
E_k \hat{\chi}(t) = A_k \hat{\chi}(t) + B_k v(t), \tag{B.23}
\]

where

\[
E_k = \begin{bmatrix}
I_k & O \\
O & \tilde{E}_f
\end{bmatrix},
\]

\[
A_k = \begin{bmatrix}
\hat{A}_k & -\hat{B}_k K_{\text{inner}} \\
O & I_{n-k} - B_k K_{\text{inner}}
\end{bmatrix},
\]

and

\[
B_k = \begin{bmatrix}
\hat{B}_k \\
\tilde{B}_f
\end{bmatrix}.
\]

The singular system in (B.23) has original \(\kappa\) finite modes and another \(q\) finite modes are induced by applying a linear preliminary feedback control law \(u(t)\) in (B.22) to the system in (B.19). All these finite modes are guaranteed to be controllable.

What follows is to decompose the singular system into a reduced-order regular system with \(\kappa + q\) controllable finite modes and nondynamic equation with \(n - \kappa - q\) infinite nondynamic ones. It can be accomplished by using previous outlined steps once again. At first, we transform the regular form into a standard one by premultiplying (B.23) by \((\gamma E_k + \eta A_k)^{-1}\), where \(\gamma\) and \(\eta\) are arbitrary scalars such that \((\gamma E_k + \eta A_k)\) is invertible. Therefore, we obtain

\[
(\gamma E_k + \eta A_k)^{-1} E_k \hat{\chi}(t) = (\gamma E_k + \eta A_k)^{-1} A_k \hat{\chi}(t) + (\gamma E_k + \eta A_k)^{-1} B_k v(t). \tag{B.24}
\]

Let

\[
\hat{\chi}(t) = \tilde{M}\tilde{\chi}(t), \tag{B.25}
\]

where the constant matrix \(\tilde{M}\) is determined by using the extended matrix sign.
Active low-order fault-tolerant state space self-tuner

function. The procedure is as the same previous illustration for finding $M$, except that it operates on $(\gamma E_k + \eta A_k)^{-1}E_k$. Substituting (B.25) into (B.24) and premultiplying it by $\tilde{M}^{-1}$ yields

$$
\tilde{M}^{-1}(\gamma E_k + \eta A_k)^{-1}E_k \tilde{M} \tilde{x}(t)
$$

$$=
\tilde{M}^{-1}(\gamma E_k + \eta A_k)^{-1}A_k \tilde{M} \tilde{x}(t) + \tilde{M}^{-1}(\gamma E_k + \eta A_k)^{-1}B_k \nu(t)
$$

$$=
\frac{1}{\eta} \left[ I_n - \gamma \tilde{M}^{-1}(\gamma E_k + \eta A_k)^{-1}E_k \tilde{M} \right] \tilde{x}(t) + \tilde{M}^{-1}(\gamma E_k + \eta A_k)^{-1}B_k \nu(t),
$$

i.e.

$$
\begin{bmatrix}
\tilde{E}_{sk} \\
\tilde{E}_{jk}
\end{bmatrix}
\begin{bmatrix}
O \\
O
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(t) = \\
\tilde{y}(t)
\end{bmatrix}
$$

$$=
\begin{bmatrix}
\frac{1}{\eta} (I_{\kappa q} - \gamma \tilde{E}_{sk}) \\
O \\
\frac{1}{\eta} (I_{n-\kappa q} - \gamma \tilde{E}_{jk})
\end{bmatrix}
\begin{bmatrix}
O \\
O
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(t) = \\
\tilde{y}(t)
\end{bmatrix}
$$

$$+
\begin{bmatrix}
\tilde{B}_{sk} \\
\tilde{B}_{jk}
\end{bmatrix}
\nu(t),
$$

where $\tilde{x}(t) = [\tilde{x}_s(t), \tilde{x}_f(t)]^T$ and $\tilde{M}^{-1}(\gamma E_k + \eta A_k)^{-1}E_k \tilde{M} = \text{block diagonal}\{\tilde{E}_{sk}, \tilde{E}_{jk}\} = \text{block diagonal}\{O_{(n-q)}, O_{(n-\kappa-q)}\}$. $\tilde{E}_{sk}$ is invertible with rank $\text{deg}\{\text{det}(sE_k - A_k)\} = (q + \kappa)$, $\tilde{E}_{jk}$ is a null matrix with dimension $(n-\kappa-q) \times (n-\kappa-q)$ and $[\tilde{B}_{sk}, \tilde{B}_{jk}]^T = \tilde{M}^{-1}(\gamma E_k + \eta A_k)^{-1}B_k$.

Moreover, premultiplying block diagonal $\{\tilde{E}_{sk}^{-1}, \frac{1}{\eta} I_{n-\kappa q}^{-1}\}$ to the above system yields

$$
\begin{bmatrix}
I_{\kappa q} \\
O
\end{bmatrix}
\begin{bmatrix}
O \\
O
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(t) = \\
\tilde{y}(t)
\end{bmatrix}
$$

$$=
\begin{bmatrix}
\frac{1}{\eta} \tilde{E}_{sk}^{-1} (I_{\kappa q} - \gamma \tilde{E}_{sk}) \\
O \\
I_{n-\kappa q}
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(t) = \\
\tilde{y}(t)
\end{bmatrix}
$$

$$+
\begin{bmatrix}
\tilde{E}_{sk}^{-1} \tilde{B}_{sk} \\
\frac{1}{\eta} I_{n-\kappa q}^{-1} \tilde{B}_{jk}
\end{bmatrix}
\nu(t),
$$

where $x_s(t) \in \mathbb{R}^{\kappa q}$, $x_f(t) \in \mathbb{R}^{n-\kappa q}$, and $\tilde{y}(t) = [\tilde{x}_s(t), \tilde{x}_f(t)]$. Thus, system (B.26) can be decomposed into the following form

$$
\tilde{x}_s(t) = A_s \tilde{x}_s(t) + B_s \nu(t),
$$

$$
\tilde{x}_f(t) = -B_f \nu(t),
$$

$$
y(t) = C, x(t) = C, MV \tilde{M} \tilde{x}(t) = C \tilde{x}(t),
$$

where $A_s = \frac{1}{\eta} \tilde{E}_{sk}^{-1} (I_{\kappa q} - \gamma \tilde{E}_{sk})$, $C = C, MV \tilde{M}$, $B_s = \tilde{E}_{sk}^{-1} \tilde{B}_{sk}$, $B_f = \frac{1}{\eta} I_{n-\kappa q}^{-1} \tilde{B}_{jk}$, and $C = C, MV \tilde{M}$.
As the same procedure (B.14) ~ (B.16), the singular system can be organized as follows

\[ y(t) = C\tilde{x}(t) = [C_1 \ C_2] \begin{bmatrix} \tilde{x}_y(t) \\ \tilde{x}_f(t) \end{bmatrix} \]

\[ = C_1\tilde{x}_y(t) + C_2\tilde{x}_f(t) \]

\[ = C_1\tilde{x}_y(t) + C_2\tilde{x}_f(t) \]

\[ = C_1\tilde{x}_y(t) + (-C_2B_f)v(t) \]

\[ = C\tilde{x}_y(t) + Dv(t), \]

(B.30)

where \( \bar{C} = C_1 \in \mathbb{R}^{p\times(k+q)} \), \( C_2 \in \mathbb{R}^{p\times(a-q-k)} \), and \( \bar{D} = -C_2B_f \in \mathbb{R}^{p\times m} \).

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