Matrices Formula for Padovan and Perrin Sequences

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Abstract

The Padovan and Perrin numbers have the matrix formula,

$$Q^n = \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

The matrix product is a $3 \times 2$ matrix that when raised to the $n^{th}$ power give a matrix product whose entries are Padovan and Perrin numbers. For which we established by mathematical induction such that,

$$Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 3 \\ P_n & P_n \\ P_{n+1} & P_{n+1} \end{bmatrix},$$

where $P_n$ and $P_n$ are the Padovan and Perrin sequences, respectively.

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1 Introduction

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay *Dom Hans van der Laan: Modern Primitive*.

In this paper, the Padovan sequence is the sequence of integers $P_n$ defined by the initial values $P_0 = 0, P_1 = 0, P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3}, \text{ for all } n \geq 3.$$
The first few values of $P_n$ are 0, 0, 1, 0, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, \ldots.

The Padovan numbers have the $Q$-matrix, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ such that
\[
Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix}, \text{ for all } n \geq 3.
\]

The Perrin sequence is the sequence of integers $P_n$ defined by a recurrence relation, and is qualitatively similar to the Lucas sequence. The initial terms are $P_0 = 3, P_1 = 0, P_2 = 2$ and subsequent terms are defined by
\[
P_n = P_{n-2} + P_{n-3}, \text{ for all } n \geq 3.
\]

Here are the first few Perrin numbers: 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, \ldots.

2 Main Results

In this study, we investigate the new property of Padovan and Perrin numbers in relation with the Padovan and Perrin matrices formula,
\[
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}.
\]

More generally, we have $Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_n & P_n \\ P_{n+1} & P_{n+1} \\ P_{n+2} & P_{n+2} \end{bmatrix}$. This strategy allows us to obtain the new relations for the Padovan and Perrin sequences.

**Theorem 2.1.** For all $n \in \mathbb{N}$ we have,
\[
Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_n & P_n \\ P_{n+1} & P_{n+1} \\ P_{n+2} & P_{n+2} \end{bmatrix}.
\]

**Proof.** Let use the principle of mathematical induction on $n$. For $n = 1$, it is easy to see that
\[
Q^1 \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^1 \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_1 & P_1 \\ P_2 & P_2 \\ P_3 & P_3 \end{bmatrix} = \begin{bmatrix} P_1 & P_1 \\ P_{1+1} & P_{1+1} \\ P_{1+2} & P_{1+2} \end{bmatrix}.
\]

Assume that it is true for all positive integer $n = k$. That is,
\[
Q^k \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^k \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} P_k & P_k \\ P_{k+1} & P_{k+1} \\ P_{k+2} & P_{k+2} \end{bmatrix}.
\]
Therefore, we have to show that it is true for \( n = k + 1 \). By the laws of associativity and exponents hold for the matrices such that their dimensions match. Consider,

\[
Q^{k+1} \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = (QQ^k) \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = Q \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_k & \mathcal{P}_k \\ P_{k+1} & \mathcal{P}_{k+1} \\ P_{k+2} & \mathcal{P}_{k+2} \end{bmatrix} = \begin{bmatrix} P_{k+1} & \mathcal{P}_{k+1} \\ P_{k+2} & \mathcal{P}_{k+2} \\ P_k + P_{k+1} & \mathcal{P}_k + \mathcal{P}_{k+1} \end{bmatrix} = \begin{bmatrix} P_{k+1} & \mathcal{P}_{k+1} \\ P_{k+2} & \mathcal{P}_{k+2} \\ P_{k+3} & \mathcal{P}_{k+3} \end{bmatrix}.
\]

Therefore, the result is true for every \( n \geq 1 \).

Let us generalize this observation using the Padovan and Perrin formula matrices.

**Proposition 2.2.** For all integers \( m, n \) such that \( 3 \leq m < n \), we have the following relations:

(a) \( P_n = P_{m-1} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1} + P_m \cdot P_{n-m+2} \).

(b) \( \mathcal{P}_n = P_{m-1} \cdot \mathcal{P}_{n-m} + P_{m+1} \cdot \mathcal{P}_{n-m+1} + P_m \cdot \mathcal{P}_{n-m+2} \).

**Proof.** From the laws of exponent for the square matrices. So, we have

\[
Q^n = Q^m Q^{n-m},
\]

it follows that

\[
Q^n \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = Q^m \left( Q^{n-m} \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \right).
\]

From the property of Padovan \( Q \)-matrix (see [2], page 2778) and Theorem 2.1 it follows that,

\[
\begin{bmatrix} P_n & \mathcal{P}_n \\ P_{n+1} & \mathcal{P}_{n+1} \\ P_{n+2} & \mathcal{P}_{n+2} \end{bmatrix} = \begin{bmatrix} P_{m-1} & P_{m+1} & P_m \\ P_m & P_{m+2} & P_{m+1} \\ P_{m+1} & P_{m+3} & P_{m+2} \end{bmatrix} \begin{bmatrix} P_{n-m} & \mathcal{P}_{n-m} \\ P_{n-m+1} & \mathcal{P}_{n-m+1} \\ P_{n-m+2} & \mathcal{P}_{n-m+2} \end{bmatrix}.
\]
yielding, upon equating corresponding elements. That is,

\[ P_n = P_{m-1} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1} + P_m \cdot P_{n-m+2}, \]

and

\[ \mathcal{P}_n = P_{m-1} \cdot \mathcal{P}_{n-m} + P_{m+1} \cdot \mathcal{P}_{n-m+1} + P_m \cdot \mathcal{P}_{n-m+2}. \]

Completes the proof.

**Remark 2.3.** In Proposition 2.2, if \( m = 3 \), then we have

\[ P_n = P_2 \cdot P_{n-3} + P_4 \cdot P_{n-2} + P_3 \cdot P_{n-1}, \]

\[ = 1 \cdot P_{n-3} + 1 \cdot P_{n-2} + 0 \cdot P_{n-1}, \text{ (replaces } P_2 = P_4 = 1 \text{ and } P_3 = 0) \]

\[ = P_{n-2} + P_{n-3}. \]

Similary, we have \( \mathcal{P}_n = \mathcal{P}_{n-2} + \mathcal{P}_{n-3} \).

**References**


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