Analytical Solution of the Frenet-Serret Systems of Circular Motion Bodies

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Abstract

In this paper, the general Frenet-Serret system of circular motion body with constant velocity is analytically solved in three dimensional space. The tangent, normal, and binormal vectors are found by reducing the system into a high order ordinary differential equation. Solving this equation gives a closed form of those vectors. A special case of four dimensional Frenet-Serret system is also solved in this work.

Keywords: Frenet-Serret, high order ODE, Tangent, Normal, Binormal, Curvature, Torsion, circular orbits

1. Introduction

The Frenet-Serret frame is one of the most important tools that analyze and describe the properties of a particle along differentiable curves in Euclidean space [1,10].
Frenet and Serret [2] adapted the frame to curves by directly expressing the changes in derivatives of the tangent, normal and binormal vectors in terms of the frame. A few decades later, after the result of Frenet and Serret, their theory was extended to surfaces [3], also an n-dimensional vector calculus formulations of the system is developed [4]. Moreover, extensions to the frame have been proposed using quaternion-formulations [1]. In applications, studying the Frenet-Serret systems is of great importance in applied mathematics, physics, engineering and many fields of science [5-19].

One of the most important applications of the Frenet-Serret frames is understanding the kinematic properties of circular bodies, like the circular orbits in black hole space [10,11]. In this case, understanding the frame is useful in studying the properties of these orbits and provides interpretation of their geometry.

The general three dimensional Frenet-Serret system to be discussed in this paper is defined by:

\[
\begin{bmatrix}
    T'(t) \\
    U'(t) \\
    V'(t)
\end{bmatrix} =
\begin{bmatrix}
    0 & \kappa_1 & \kappa_2 \\
    -\kappa_1 & 0 & \tau \\
    -\kappa_2 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
    T(t) \\
    U(t) \\
    V(t)
\end{bmatrix}
\]

Where \( T, U, V \) are the tangent, normal and binormal vector fields respectively, \( t \) is the time, \( \kappa_1 = T' \bullet U, \kappa_2 = T' \bullet V \), and \( \tau \) is the torsion.

Studying of systems like (1) has been carried out in both analytical and numerical approaches as in [20-25]. This System will be analytically solved in this paper for bodies of circular motion with constant velocities.

2. Analysis and results

2.1 The Three Dimensional System

Considering a circular motion body with constant velocity leads to a constant curvature and torsion, hence, differentiating the third equation of (1) twice and differentiating the first and the second equations once with respect to time give

\[ V'''' = -\kappa_2 T'' - \tau U'' \]

(2)
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\[ T'' = \kappa_1 U' + \kappa_2 V' \]  \hfill (3)

and

\[ U'' = -\kappa_1 T' + \tau V'. \]  \hfill (4)

Substituting (3) and (4) into (2) gives:

\[ V''' = -\kappa_2 (\kappa_1 U' + \kappa_2 V') - \tau (-\kappa_1 T' + \tau V') \]  \hfill (5)

which is written as:

\[ V''' = \kappa_1 \tau T' - \kappa_1 \kappa_2 U' - (\kappa_2^2 + \tau^2) V' \]  \hfill (6)

But

\[ T' = \kappa_1 U + \kappa_2 V \]  \hfill (7)

and

\[ U' = -\kappa_1 T + \tau V. \]  \hfill (8)

Therefore, substituting (7) and (8) in (6) yields to

\[ V''' = \kappa_1^2 \kappa_2 T + \kappa_1^2 \tau U + (-\kappa_2^2 - \tau^2) V' \]  \hfill (9)

hence,

\[ V''' = \kappa_1^2 (\kappa_2 T + \tau U) - (\kappa_2^2 + \tau^2) V' \]  \hfill (10)

but from (1),

\[ \kappa_2 T + \tau U = -V'. \]  \hfill (11)

Substituting (11) in (10) gives

\[ V''' = -\kappa_1^2 V' - (\kappa_2^2 + \tau^2) V' \]  \hfill (12)

which is

\[ V''' + (\kappa_1^2 + \kappa_2^2 + \tau^2) V' = 0 \]  \hfill (13)

The characteristic equation of the homogeneous ordinary differential equation (13) is

\[ r^3 + (\kappa_1^2 + \kappa_2^2 + \tau^2) r = 0. \]  \hfill (14)

In addition to the trivial solution, The solution of (14) is

\[ r = \pm i \sqrt{\kappa_1^2 + \kappa_2^2 + \tau^2}, \]  \hfill where \( i = \sqrt{-1} \), hence, the solution of \( V \) is

\[ V(t) = C_1 + C_2 \cos(\sqrt{\kappa_1^2 + \kappa_2^2 + \tau^2} t) + C_3 \sin(\sqrt{\kappa_1^2 + \kappa_2^2 + \tau^2} t) \]  \hfill (15)

Now, as \( V \) is known, the following system has to be solved for \( T \) and \( U \),

\[ T' = \kappa_1 U + \kappa_2 V \]

\[ U' = -\kappa_1 T + \tau V \]  \hfill (16)

From (16)
\[ U'' = -\kappa_1 T' + \tau V' \]  

Substituting the first equation of (16) into (17) gives
\[ U'' = -\kappa_1^2 U - \kappa_1 \kappa_2 V + \tau V' \]  

which is
\[ U'' + \kappa_1^2 U = F(t) \]  

where
\[ F(t) = -\kappa_1 \kappa_2 C_1 + \kappa_3 C_4 \sin(\alpha t) + \kappa_3 C_4 \cos(\alpha t) \]  

If \( \alpha = \sqrt{\kappa_1^2 + \kappa_2^2 + \tau^2} \), then
\[ F(t) = -\kappa_1 \kappa_2 C_1 + (\kappa_1 \kappa_2 \kappa_3) \cos(\alpha t) - (\kappa_1 \kappa_2 \kappa_3) \sin(\alpha t) \]  

Using the variation of parameters method, the solution of (19) is
\[ U = -\kappa_1 C_1 + A_1 \cos(\alpha t) + A_1 \sin(\alpha t) + \frac{\kappa_1 \kappa_2 \kappa_3}{\kappa_1^2 + \tau^2} \sin(\alpha t) + \frac{\kappa_1 \kappa_2 \kappa_3}{\kappa_2^2 + \tau^2} \cos(\alpha t) \]  

Now, as \( U \) and \( V \) are known, finding \( T \) is obvious by solving the first equation of (16).

2.2 The four Dimensional System

Consider the following well-known four dimensional Frenet-Serret system [4]:

\[
\begin{bmatrix}
T' \\
U' \\
V' \\
W'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa_1 & 0 & 0 \\
0 & 0 & \kappa_2 & 0 \\
0 & -\kappa_2 & 0 & \kappa_3 \\
0 & 0 & -\kappa_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
U \\
V \\
W
\end{bmatrix}.
\]

(23)

It is clear that
\[ U'' = -\kappa_1 T' + \kappa_2 V' \]  

and
\[ U^{(4)} = -\kappa_1 T^{(3)} + \kappa_2 V^{(3)} \]  

So
\[ U^{(4)} = -\kappa_1 (U'') + \kappa_2 (-\kappa_2 U'' + \kappa_3 W'') \]  

And hence,
\[ U^{(4)} = -\kappa_1^2 U'' - \kappa_2^2 U'' + \kappa_2 \kappa_3 W'' \]  

But \( W'' = -\kappa_3 V' \). Hence, from from (24),
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\[ V' = \frac{1}{k_2} [U'' + k_1 T'] \quad (28) \]

So

\[ W'' = -k_3 \left[ \frac{1}{k_2} (U'' + k_1 T') \right] \quad (29) \]

But \( T' = k_1 U \), so

\[ W'' = -\frac{k_3}{k_2} [U'' + k_1 k_1 U] \quad (30) \]

So

\[ U^{(4)} = -k_1^2 U'' - k_2^2 U'' + k_2 k_3 \left( \frac{k_3}{k_2} [U'' + k_1^2 U] \right) \quad (31) \]

therefore

\[ U^{(4)} + \left( k_1^2 + k_2^2 + k_3^2 \right) U'' + k_1 k_2 k_3 U = 0 \quad (32) \]

Equation (32) is a homogeneous fourth order ordinary differential equations. Solving it for \( U \) makes the solution of system (23) obvious.

Conclusions and Future Perspectives

In this paper, the Frenet-Serret system (1) is efficiently reduced to a homogeneous third order ordinary differential equation which is solved for the binormal vector field. The normal vector field is obtained by solving a linear system of first order ordinary differential equations, while the tangent vector field can be found by solving a simple linear ordinary differential equation. A special case of four dimensional Frenet-Serret system when the torsion is zero has been analytically solved. As a next step, a circular motion bodies with non-constant velocities will be under consideration.

References


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