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Multi-objective Programming Approach for Fuzzy Linear Programming Problems

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Abstract

A new method namely, level-sum method based on the multi-objective linear programming and the simplex method is proposed for computing an optimal fuzzy solution to a fuzzy linear programming problem inwhich fuzzy ranking functions are not used. The level-sum method is illustrated by numerical examples.

Keywords: Fuzzy linear programming problem, Optimal fuzzy solution, Multi-objective linear programming, Level-sum method.

1. Introduction

Linear programming (LP) is one of the most applicable optimization techniques. It deals with the optimizations of a linear function while satisfying a set of linear equality and/or inequality constraints or restrictions. In practice, a LP model involves a lot of parameters whose values are assigned by experts / decision makers. However, both experts and decision makers do not precisely know the value of those parameters in most of the cases. Therefore, fuzzy linear programming (FLP) problem [9,2] was introduced and studied. In the literature, a variety of algorithms for solving FLP have been studied based on fuzzy ranking function and classical linear programming. Tanaka et al. [7], Zimmerman [10], Buckley and Feuring [3], Thakre et al.[8] and Zhang et al. [11] solved FLP problems using multi-objective linear programming (MOLP) technique. Pandian[6] has proposed a new approach, namely sum of objectives (SO) method for finding a properly efficient solution to multi-objective programming problems.

In this paper, we propose a new method namely, level-sum method for finding an optimal fuzzy solution to fully FLP problems which is a crisp LP technique not using the fuzzy ranking function. First, we construct a crisp MOLP problem from the FLP problem and then, we establish a relation between an optimal fuzzy solution to the fully FLP problem and an efficient solution to its related MOLP problem. Based on the relation, we develop the proposed method. With the help of numerical examples, the level-sum method is illustrated. The advantages of the proposed method are that fuzzy ranking functions are not used, the obtained results exactly satisfy all the constraints and the computation can be made by LP solver because it is based only on crisp LP technique.

2. Preliminaries

We need the following definitions of the basic arithmetic operators and partial ordering relations on fuzzy triangular numbers based on the function principle which can be found in [2,4, 9,10].

Definition 2.1 A fuzzy number $\tilde{a}$ is a triangular fuzzy number denoted by (a_1, a_2, a_3) where a_1, a_2 and a_3 are real numbers and its member ship function $\mu_{\tilde{a}}(x)$ is given below:

$$\mu_{\tilde{a}}(x) = \begin{cases} (x - a_1)/(a_2 - a_1) & \text{for } a_1 \le x \le a_2 \\ (a_3 - x)/(a_3 - a_2) & \text{for } a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases}$$

Let $F(R)$ be the set of all real triangular fuzzy numbers.

Definition 2.2 Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be in $F(R)$. Then,

(i) $(a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

(ii) $(a_1, a_2, a_3) \,\Theta\, (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.

(iii) $k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3)$, for $k \ge 0$.

(iv) $k(a_1, a_2, a_3) = (ka_3, ka_2, ka_1)$, for $k < 0$.

(v) $$(a_1, a_2, a_3) \otimes (b_1, b_2, b_3) = \begin{cases} (a_1b_1, a_2b_2, a_3b_3), & a_1 \ge 0, \\ (a_1b_3, a_2b_2, a_3b_3), & a_1 < 0, a_3 \ge 0, \\ (a_1b_3, a_2b_2, a_3b_1), & a_3 < 0. \end{cases}$$

Definition 2.3 Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be in $F(R)$, then

(i) $\tilde{A} \approx \tilde{B}$ iff $a_i = b_i, i = 1,2,3$; (ii) $\tilde{A} \preceq \tilde{B}$ iff $a_i \le b_i, i = 1,2,3$ and

(iii) $\tilde{A} \succeq \tilde{B}$ iff $a_i \ge b_i, i = 1,2,3$.

Based on the notations of Mangasarian [5], we define the following partial order relation '$\succ$' as follows.

Definition 2.4 Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be in $F(R)$, then $\tilde{A} \succ \tilde{B}$ iff $a_i \geq b_i$, $i = 1,2,3$ and $a_r > b_r$, for some $r \in \{1,2,3\}$.

Consider the following multi-objective optimization problem

(MP) Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$

subject to $g(x) \leqq 0$, $x \in X$,

where $f_i : X \to R$, $i = 1,2,\ldots,k$ and $g : X \to R^m$ where $g = (g_1, \ldots, g_m)$ are differentiable functions on X, an open convex subset of R^n.

Now, $P = \{x \in X : g_j(x) \leq 0, j = 1,2,\ldots m\}$ is the set of all feasible solutions for the problem (MP).

Definition 2.5: A feasible point x° is said to be efficient [6] for (MP) if there exists no other feasible point x in P such that $f_i(x) \leq f_i(x^\circ)$, $i = 1,2,\ldots,k$ and $f_r(x) < f_r(x^\circ)$, for some $r \in \{1,2,\ldots,k\}$.

3. Fully Fuzzy Linear Programming Problem

Consider the following fully FLP with m fuzzy inequality/equality constraints and n fuzzy variables may be formulated as follows:

(P) Maximize $\tilde{z} \approx \tilde{c}^T \tilde{x}$

subject to $\tilde{A} \otimes \tilde{x}$ $\{\preceq, \approx, \succeq\}$ $\tilde{b}$, $\tilde{x} \succeq \tilde{0}$,

where $\tilde{a}_{ij}, \tilde{c}_j, \tilde{x}_j, \tilde{b}_i \in F(R)$, for all $1 \leq j \leq n$ and $1 \leq i \leq m$, $\tilde{c}^T = (\tilde{c}_j)_{1\times n}$, $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$, $\tilde{x} = (\tilde{x}_j)_{nx1}$ and $\tilde{b} = (\tilde{b}_i)_{mx1}$.

Let the parameters $\tilde{z}$, $\tilde{a}_{ij}, \tilde{c}_j, \tilde{x}_j$ *and* $\tilde{b}_i$ be the triangular fuzzy numbers (z_1, z_2, z_3), (p_j, q_j, r_j), (x_j, y_j, t_j), (a_{ij}, b_{ij}, c_{ij}) and (b_i, g_i, h_i) respectively. Then, the problem (P) can be written as follows:

(P) Maximize $(z_1, z_2, z_3) \approx \sum_{j=1}^{n} (p_j, q_j, r_j) \otimes (x_j, y_j, t_j)$

subject to

$$\sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, t_j) \{\preceq, \approx, \succeq\} (b_i, g_i, h_i), \text{ for all } i = 1,2,\ldots,m$$

$$(x_j, y_j, t_j) \succeq \tilde{0}, \ j = 1,2,\ldots,m.$$

Now, using the arithmetic operations and partial ordering relations, we write the given FLPP as a MOLP problem which is given below:

(M) Maximize $z_1 = \sum_{j=1}^{n}$ lower value of $\left((p_j, q_j, r_j) \otimes (x_j, y_j, t_j)\right)$

Maximize $z_2 = \sum_{j=1}^{n}$ middle value of $\left((p_j, q_j, r_j) \otimes (x_j, y_j, t_j)\right)$

Maximize $z_3 = \sum_{j=1}^{n}$ upper value of $\left((p_j, q_j, r_j) \otimes (x_j, y_j, t_j)\right)$

subject to

$\sum_{j=1}^{n}$ lower value of $\left((a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, t_j)\right)\{\leq, =, \geq\} b_i$, for all $i = 1,2,...,m$;

$\sum_{j=1}^{n}$ middle value of $\left((a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, t_j)\right)\{\leq, =, \geq\} g_i$, for all $i = 1,2,...,m$;

$\sum_{j=1}^{n}$ upper value of $\left((a_{ij}, b_{ij}, c_{ij}) \otimes (x_j, y_j, t_j)\right)\{\leq, =, \geq\} h_i$, for all $i = 1,2,...,m$;

$z_2 \geq z_1$; $z_3 \geq z_2$; $x_j \leq y_j$, j =1,2,…,m ; $y_j \leq t_j$, j =1,2,…,m

$x_j \geq 0$, j =1,2,…,m.

Remark 3.1: In the case of a fully FLP problem involving trapezoidal fuzzy numbers and / or trapezoidal fuzzy decision variables, we get a MOLP problem having four objectives.

We, now prove the following theorem which establish a relation between an optimal fuzzy solution to a fully FLP problem and an efficient solution to its related MOLP problems.

Theorem 3.1: Let $X^\circ = \{ x_j^\circ, y_j^\circ, t_j^\circ$; j =1,2,…,m} be an efficient solution to the problem (M). Then, $\tilde{X}^\circ = \{ (x_j^\circ, y_j^\circ, t_j^\circ)$; j =1,2,…,m} is an optimal solution to the problem (P).

Proof: Now, since $\tilde{X}^\circ = \{ x_j^\circ, y_j^\circ, t_j^\circ$; j =1,2,…,m} is an efficient solution to the problem (M), $\tilde{X}^\circ = \{ (x_j^\circ, y_j^\circ, t_j^\circ)$; j =1,2,…,m} is a feasible solution to the problem (P).

Assume that $\tilde{X}^\circ = \{ (x_j^\circ, y_j^\circ, t_j^\circ)$; j =1,2,…,m} is not optimal to the problem (P). Then, there exists a feasible solution $\tilde{X} = \{ (x_j, y_j, t_j)$; j =1,2,…,m} to the problem (P) such that $Z(\tilde{X}) \succ Z(\tilde{X}^\circ)$, that is, $z_i(x, y, t) \geq z_i(x^\circ, y^\circ, t^\circ)$, $i = 1,2,3$ and

$z_r(x,y,t) > z_r(x^\circ, y^\circ, t^\circ)$, for some $r \in \{1,2,3\}$ where $x^\circ = \{x_j^\circ;$ j =1,2,…,m}, $y^\circ = \{y_j^\circ;$ j =1,2,…,m}, $t^\circ = \{t_j^\circ;$ j =1,2,…,m}, $x = \{x_j;$ j =1,2,…,m}, $y = \{y_j;$ j =1,2,…,m} and $t = \{t_j;$ j =1,2,…,m}. This means that $X^\circ = \{x_j^\circ, y_j^\circ, t_j^\circ;$ j =1,2,…,m} is not an efficient solution to the problem (M) which is a contradiction. Hence the theorem is proved.

Now, we propose a new method namely, level-sum method for finding an optimal fuzzy solution to a FLP problem which is based on MOLP and the simplex method.

The proposed method proceeds as follows:

Step 1: Construct a crisp MOLP problem from the given FLP problem.
Step 2: Find an efficient solution to the MOLP problem obtained in the Step 1. using the SO method [6].
Step 3: The efficient solution obtained form the Step 2. to the MOLP problem yields an optimal fuzzy solution to the FLP problem by the Theorem 3.1..

Remark 3.2: The proposed method can be extended to fuzzy integer LP problems by adding the integer restrictions and replacing the simplex method by an integer LP technique.

The proposed method is illustrated by the following examples.

Example 3.1: Consider the following fully FLP problem:

Maximize $\tilde{z} \approx (-1, 2, 3) \otimes \tilde{x}_1 \oplus (2, 3, 4) \otimes \tilde{x}_2$

subject to

$(0,1,2) \otimes \tilde{x}_1 \oplus (1,2,3) \otimes \tilde{x}_2 \approx (2,10,24);$

$(1,2,3) \otimes \tilde{x}_1 \oplus (0,1,2) \otimes \tilde{x}_2 \approx (1,8,21);$ $\tilde{x}_1$ and $\tilde{x}_2 \succeq \tilde{0}$.

Let $\tilde{x}_1 = (x_1, y_1, t_1)$, $\tilde{x}_2 = (x_2, y_2, t_2)$ and $\tilde{z} = (z_1, z_2, z_3)$.

Now, using the Step 1., the MOLP problem related to the given fully FLP problem is given below:

(M) Maximize $z_1 = -t_1 + 2x_2$

Maximize $z_2 = 2y_1 + 3y_2$

Maximize $z_3 = 3t_1 + 4t_2$

subject to

$0x_1 + x_2 = 2;$ $y_1 + 2y_2 = 10;$ $2t_1 + 3t_2 = 24;$ $x_1 + 0x_2 = 1;$

$2y_1 + y_2 = 8;$ $3t_1 + 2t_2 = 21;$ $z_2 \geq z_1;$ $z_3 \geq z_2;$ $y_1 \geq x_1;$ $y_2 \geq x_2;$

$t_1 \geq y_1;$ $t_2 \geq y_2;$ $x_1, x_2 \geq 0$.

Now, by the Step 2., we consider the following LP problem (S) related to the above MOLP problem:

(S) Maximize $Z = 2x_2 + 2y_1 + 3y_2 + 2t_1 + 4t_2$

subject to

$$0x_1 + x_2 = 2\,;\; y_1 + 2y_2 = 10\,;\, 2t_1 + 3t_2 = 24\,;\, x_1 + 0x_2 = 1\;;\; 2y_1 + y_2 = 8\,;$$
$$3t_1 + 2t_2 = 21\,;\; -2x_2 + 2y_1 + 3y_2 + t_1 \ge 0\,;\quad -2y_1 - 3y_2 + 3t_1 + 4t_2 \ge 0\,;$$
$$y_1 - x_1 \ge 0\,;\quad y_2 - x_2 \ge 0\,;\; t_1 - y_1 \ge 0\,;\; t_2 - y_2 \ge 0\;;\; x_1\,,\, x_2 \ge 0$$

and solve it by simplex method. The optimal solution to the problem (S) is

$$x_1 = 1\;;\; x_2 = 2\;;\; y_1 = 2\,;\; y_2 = 4\,;\; t_1 = 3 \;\text{ and }\; t_2 = 6 \;\text{ with }\; Z = 50\,.$$

Thus, ($x_1 = 1$, $x_2 = 2$, $y_1 = 2$, $y_2 = 4$, $t_1 = 3$, $t_2 = 6$) is an efficient solution to the problem (M).

Now, by the Step 3., $\tilde{x}_1 \approx (1, 2, 3)$, $\tilde{x}_2 \approx (2, 4, 6)$ and $\tilde{z} \approx (1, 16, 33)$ is an optimal fuzzy solution to the given fully FLP problem.

Remark 3.3 : For the fully FLP problem (the Example 4.1), Amit Kumar et al. [1] by the ranking method obtained the same optimal fuzzy solution.

Example 3.2: Consider the following FLP problem:

Maximize $\tilde{z} \approx (7, 10, 14, 25) \otimes x_1 \oplus (20, 25, 35, 40) \otimes x_2$

subject to

$$(1, 3, 4) \otimes x_1 \oplus (2, 6, 7) \otimes x_2 \preceq (8, 13, 15);$$
$$(3, 4, 6) \otimes x_1 \oplus (1, 6, 10) \otimes x_2 \preceq (3, 7, 9);\; x_1 \text{ and } x_2 \ge 0\,.$$

Let $\tilde{z} = (z_1, z_2, z_3, z_4)$. Now, using the Step 1., the MOLP problem related to the given FLP problem is given below:

(M) Maximize $z_1 = 7x_1 + 20x_2$

Maximize $z_2 = 10x_1 + 25x_2$

Maximize $z_3 = 14x_1 + 35x_2$

Maximize $z_4 = 25x_1 + 40x_2$

subject to

$$x_1 + 2x_2 \le 8\,;\; 3x_1 + 6x_2 \le 13\,;\; 4x_1 + 7x_2 \le 15\,;\; 3x_1 + x_2 \le 3$$
$$4x_1 + 6x_2 \le 7\,;\; 6x_1 + 10x_2 \le 9\,;\; z_2 \ge z_1\,;\; z_3 \ge z_2\,;\; z_4 \ge z_3\;;\; x_1\,,\, x_2 \ge 0\,.$$

Now, by the Step 2., we consider the following LP problem (S) related to the above MOLP problem:

(S) Maximize $Z = 56x_1 + 120x_2$

subject to

$$x_1 + 2x_2 \le 8\,;\; 3x_1 + 6x_2 \le 13\,;\; 4x_1 + 7x_2 \le 15\,;\; 3x_1 + x_2 \le 3\;\; 4x_1 + 6x_2 \le 7\,;$$
$$6x_1 + 10x_2 \le 9\,;\; 3x_1 + 5x_2 \ge 0\,;\; 4x_1 + 10x_2 \ge 0\,;\; 9x_1 + 5x_2 \ge 0\;;\; x_1\,,\, x_2 \ge 0$$

and solve it by simplex method. The optimal solution to the problem (S) is

$$x_1 = 0 \;\text{ and }\; x_2 = 0.9 \;\text{ with }\; Z = 108\,.$$

Thus, ($x_1 = 0$, $x_2 = 0.9$) is an efficient solution to the problem (M).

Now, by the Step 3., $x_1 = 0$, $x_2 = 0.9$ and $\tilde{z} \approx (18, 25.5, 31.5, 36)$ is an optimal fuzzy solution to the given FLP problem.

Remark 3.4 : In Thakre et al. [8] by the weighted method, the same optimal fuzzy solution to the FLP problem (the Example 3.2.) is obtained.

4. Conclusion

In this paper, we propose the level-sum method to find an optimal fuzzy solution to a FLP problem satisfying all constraints. The main advantage of the proposed method is that the FLP problems can be solved by any LP solver using the level-sum method since it is based on only simplex method. The level-sum method can serve managers by providing an appropriate best solution to a variety of LP models having fuzzy numbers and variables in a simple and effective manner. In near future, we extend the level-sum method to fuzzy MOLP problems.

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