On the Twisted $q$-Tangent Numbers and Polynomials

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Abstract

In this paper we introduce the twisted $q$-tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$. Some interesting results and relationships are obtained.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Euler numbers and polynomials, tangent numbers and polynomials, twisted $q$-tangent numbers and polynomials

1 Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g|g: \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\},$$
the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1} = \lim_{N \to \infty} \sum_{0 \leq x < p^N} g(x)(-1)^x, \quad \text{(see [1])}. \quad (1.1)$$

If we take $g_1(x) = g(x + 1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \quad \text{(see [1-4])}. \quad (1.2)$$

From (1.1), we obtain

$$\int_{\mathbb{Z}_p} g(x + n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.3)$$

Let us define the tangent numbers $T_n$ and polynomials $T_n(x)$ as follows:

$$\int_{\mathbb{Z}_p} e^{2yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}, \quad \text{(see [3])}. \quad (1.4)$$

$$\int_{\mathbb{Z}_p} e^{(x+2y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \quad (1.5)$$

Numerous properties of tangent number are known. More studies and results in this subject we may see references [3], [4], [5]. About extensions for the tangent numbers can be found in [5].

Recently, many mathematicians have studied in the area of the $q$-analogue of the Bernoulli numbers, Euler numbers, and Genocchi numbers (see [1-5]). Our aim in this paper is to define twisted $q$-tangent polynomials $T_{n,q,w}(x)$. We investigate some properties which are related to twisted $q$-tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$. We also derive the existence of a specific interpolation function which interpolate twisted $q$-tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$ at negative integers.

2 Twisted $q$-tangent numbers and polynomials

Our primary goal of this section is to define twisted $q$-tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$. We also find generating functions of twisted $q$-tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$ and investigate their properties. Let $T_p = \bigcup_{N \geq 1} C_{pN} = \lim_{N \to \infty} C_{pN}$, where $C_{pN} = \{w \mid w^{pN} = 1\}$ is the cyclic group of order $p^N$. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally
constant function $x \mapsto w^x$. For $w \in T_p$ and $q \in \mathbb{C}_p$ with $|1 - q|_p \leq 1$, if we take $g(x) = q^x \phi_w(x)e^{2xt}$ in (1.2), then we easily see that

$$I_{-1}(q^x \phi_w(x)e^{2xt}) = \int_{\mathbb{Z}_p} q^x \phi_w(x)e^{2xt}d\mu_{-1}(x) = \frac{2}{wqe^{2t} + 1}.$$ 

Let us define the twisted $q$-tangent numbers $T_{n,q,w}$ and polynomials $T_{n,q,w}(x)$ as follows:

$$I_{-1}(q^y \phi_w(y)e^{2yt}) = \int_{\mathbb{Z}_p} q^y \phi_w(y)e^{2yt}d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q,w} \frac{t^n}{n!}, \quad (2.1)$$

$$I_{-1}(q^y \phi_w(y)e^{(2y+x)t}) = \int_{\mathbb{Z}_p} q^y \phi_w(y)e^{(x+2y)t}d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q,w}(x) \frac{t^n}{n!}. \quad (2.2)$$

By (2.1) and (2.2), we obtain the following Witt’s formula.

**Theorem 2.1** For $n \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} q^x \phi_w(x)(2x)^n d\mu_{-1}(x) = T_{n,q,w},$$

$$\int_{\mathbb{Z}_p} q^y \phi_w(y)(x+2y)^n d\mu_{-1}(y) = T_{n,q,w}(x).$$

By using $p$-adic integral on $\mathbb{Z}_p$, we obtain,

$$\int_{\mathbb{Z}_p} q^x \phi_w(x)e^{2xt}d\mu_{-1}(x) = 2 \sum_{m=0}^{\infty} (-1)^m w^m q^m e^{2mt}. \quad (2.3)$$

Thus twisted $q$-tangent numbers $T_{n,q,w}$ are defined by means of the generating function

$$F_{q,w}(t) = \sum_{n=0}^{\infty} T_{n,q,w} \frac{t^n}{n!} = 2 \sum_{m=0}^{\infty} (-1)^m w^m q^m e^{2mt}. \quad (2.4)$$

Using similar method as above, by using $p$-adic integral on $\mathbb{Z}_p$, we have

$$\sum_{n=0}^{\infty} T_{n,q,w}(x) \frac{t^n}{n!} = \left( \frac{2}{wqe^{2t} + 1} \right) e^{xt}. \quad (2.5)$$

By using (2.2) and (2.5), we have

$$F_{q,w}(t, x) = \sum_{n=0}^{\infty} T_{n,q,w}(x) \frac{t^n}{n!} = 2 \sum_{m=0}^{\infty} (-1)^m w^m q^m e^{(2m+x)t}. \quad (2.6)$$
By Theorem 2.1, we easily obtain that
\[
T_{n,q,w}(x) = \int_{\mathbb{Z}_p} q^y \phi_w(y)(x + 2y)^n d\mu_{-1}(y)
= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} T_{k,q,w}
= 2 \sum_{m=0}^{\infty} (-1)^m w^m q^m (x + 2m)^n.
\]

The following elementary properties of tangent polynomials \(T_{n,q,w}(x)\) are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [1]-[4].

\textbf{Theorem 2.2} For any positive integer \(n\), we have
\[
T_{n,q,w}(x) = (-1)^n w^{-1} q^{-1} T_{n,q^{-1},w^{-1}}(2 - x).
\]

\textbf{Theorem 2.3} For any positive integer \(m (=\text{odd})\), we have
\[
T_{n,q,w}(x) = m^n \sum_{a=0}^{m-1} (-1)^a w^a q^a T_{n,q^a,w^m} \left( \frac{2a + x}{m} \right), \quad n \in \mathbb{Z}_+.
\]

By (1.3), (2.1), and (2.2), we easily see that
\[
2^{m+1} \sum_{l=0}^{n-1} (-1)^{n-1-l} w^l q^l m^l = w^n q^n T_{m,q,w}(2n) + (-1)^{n-1} T_{m,q,w}.
\]

Hence, we have the following theorem.

\textbf{Theorem 2.4} Let \(m \in \mathbb{Z}_+\). If \(n \equiv 0 \pmod{2}\), then
\[
w^n q^n T_{m,q,w}(2n) - T_{m,q,w} = 2^{m+1} \sum_{l=0}^{n-1} (-1)^{l+1} w^l q^l m^l.
\]

If \(n \equiv 1 \pmod{2}\), then
\[
w^n q^n T_{m,q,w}(2n) + T_{m,q,w} = 2^{m+1} \sum_{l=0}^{n-1} (-1)^l w^l q^l m^l.
\]
From (1.3), we note that
\[
2 =qw \int_{\mathbb{Z}_p} q^x \phi_w(x)e^{(2x+2)t}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x \phi_w(x)e^{2xt}d\mu_{-1}(x)
\]
\[
= \sum_{n=0}^{\infty} \left( \left[ wq \int_{\mathbb{Z}_p} q^x \phi_w(x)(2x + 2)^n d\mu_{-2}(x) + \int_{\mathbb{Z}_p} q^x \phi_w(x)(2x)^n d\mu_{-1}(x) \right] \frac{t^n}{n!} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( wqT_{n,q,w}(2) + T_{n,q,w} \right) \frac{t^n}{n!}.
\]

Therefore, we have the following theorem.

**Theorem 2.5** For \( n \in \mathbb{Z}_+ \), we have

\[
wqT_{n,q,w}(2) + T_{n,q,w} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}
\]

By (2.7) and Theorem 2.5, we have the following corollary.

**Corollary 2.6** For \( n \in \mathbb{Z}_+ \), we have

\[
wq(T_{q,w} + 2)^n + T_{n,q,w} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}
\]

with the usual convention of replacing \((T_{q,w})^n\) by \(T_{n,q,w}\).

**Theorem 2.7** For \( n \in \mathbb{Z}_+ \), we have

\[
T_{n,q,w}(x + y) = \sum_{k=0}^{n} \binom{n}{k} T_{k,q,w}(x)y^{n-k}.
\]

By Theorem 2.1, we easily get

\[
T_{n,q,w}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} q^y \phi_w(y)(2y)^l d\mu_{-1}(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} T_{l,q,w}.
\]

Therefore, we obtain the following theorem.

**Theorem 2.8** For \( n \in \mathbb{Z}_+ \), we have

\[
T_{n,q,w}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q,w}x^{n-l}.
\]
3 The twisted $q$-tangent zeta function

In this section, by using twisted $q$-tangent numbers and polynomials, we give the definition for the twisted $q$-tangent zeta function and Hurwitz-type twisted $q$-tangent zeta functions. These functions interpolate the twisted $q$-tangent numbers and tangent polynomials, respectively. Let $q$ be a complex number with $|q| < 1$ and $w$ be the $p^N$-th root of unity. From (2.4), we note that
\[
\frac{d^k}{dt^k} F_{q,w}(t) \bigg|_{t=0} = 2 \sum_{m=0}^{\infty} (-1)^m w^m q^m (2m)^k = T_{k,q,w}, \quad (k \in \mathbb{N}).
\]  

By using the above equation, we are now ready to define twisted $q$-tangent zeta functions.

**Definition 3.1** Let $s \in \mathbb{C}$ with $\text{Re}(s) > 0$.
\[
\zeta_{q,w}(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n w^n q^n}{(2n)^s}.
\]  

Note that $\zeta_{q,w}(s)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_{q,w}(s)$ and $T_{k,q,w}$ is given by the following theorem.

**Theorem 3.2** For $k \in \mathbb{N}$, we have
\[
\zeta_{q,w}(-k) = T_{k,q,w}.
\]

Observe that $\zeta_{q,w}(s)$ function interpolates $T_{k,q,w}$ numbers at non-negative integers. By using (2.7), we note that
\[
\frac{d^k}{dt^k} F_{q,w}(t, x) \bigg|_{t=0} = 2 \sum_{m=0}^{\infty} (-1)^m w^m q^m (x + 2m)^k = T_{k,q,w}(x), \quad (k \in \mathbb{N}),
\]
and
\[
\left( \frac{d}{dt} \right)^k \left( \sum_{n=0}^{\infty} T_{n,q,w}(x) \frac{t^n}{n!} \right) \bigg|_{t=0} = T_{k,q,w}(x), \quad \text{for } k \in \mathbb{N}.
\]  

By (3.2) and (3.4), we are now ready to define the Hurwitz-type twisted $q$-tangent zeta functions.

**Definition 3.3** Let $s \in \mathbb{C}$ with $\text{Re}(s) > 0$.
\[
\zeta_{q,w}(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n w^n q^n}{(2n + x)^s}.
\]  

Note that $\zeta_{q,w}(s, x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_{q,w}(s, x)$ and $T_{k,q,w}(x)$ is given by the following theorem.

**Theorem 3.4** For $k \in \mathbb{N}$, we have
\[
\zeta_{q,w}(-k, x) = T_{k,q,w}(x).
\]
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References


Received: July 5, 2013