Applied Mathematical Sciences, vol. 8, 2014, no. 147, 7341 - 7357
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.49740

Statistical Characteristics of the Magnitude and Location of the Greatest Maximum of Markov Random Process with Piecewise Constant Drift and Diffusion Coefficients

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Abstract

We considered a method for the determining of the statistical characteristics of the magnitude, location and first-passage time of Markov random process, with piecewise constant drift and diffusion coefficients. We found the closed analytical expressions for distribution functions of the specified random variables. We also analyzed the asymptotic behavior of probability density and ordinary moments of location of the greatest maximum of Markov random process and showed their coincidence with some known results for the particular cases.

Keywords: Markov random process, greatest maximum of random process, first-passage time, limiting statistical characteristics, piecewise constant drift and diffusion coefficients, discontinuous signal detection and measuring, local Markov approximation method

1 Statement problem

In a number of practical appendices of the information theory it is necessary to define the magnitude (for example, for detection of a useful signal in the observable data realization) and location (for example, for estimation of informative parameter on the observable data realization) of the greatest maximum of Markov random process of diffusion type with piecewise constant drift and diffusion coefficients and also to calculate their statistical characteristics [1-6]. Often such problem takes place in the statistical analysis of the quasi-deterministic or random processes, some characteristics of which are described by step functions, against hindrances. In the literature it is accepted to name such processes as discontinuous [1-3].

So, let the Markov random process $y(x)$, defined on the interval $x \in [X_1, X_2]$, possesses drift coefficient

$$K_1 = \begin{cases} a_1, & X_1 \le x \le x_0, \\ -a_2, & x_0 < x \le X_2, \end{cases} \tag{1}$$

and diffusion coefficient

$$K_2 = \begin{cases} b_1, & X_1 \le x \le x_0, \\ b_2, & x_0 < x \le X_2. \end{cases} \tag{2}$$

Here a_i, b_i, $i = 1,2$ are some positive constants, x_0 is interior point of the segment $[X_1, X_2]$.

Let us designate as

$$Y = \sup_{X_1 \le x \le X_2} y(x), \qquad x_m = \arg\sup_{X_1 \le x \le X_2} y(x) \tag{3}$$

the magnitude and location of the greatest maximum of random process $y(x)$ in the segment $[X_1, X_2]$ accordingly. It is necessary to find statistical characteristics (distribution functions, probability densities and moments) of the random variables Y and x_m (3).

2 Probability Density of Magnitude and Location of the Greatest Maximum of Markov Random Process with Piecewise Constant Drift and Diffusion Coefficients

In order to find statistical characteristics of random variables Y and x_m (3), we put into consideration two-dimensional distribution function of the greatest maximum magnitude of process $y(x)$:

$$F_2(u,v,X) = P\left[\sup_{X_1 \le x \le X} y(x) < u, \sup_{X < x < X_2} y(x) < v\right]. \tag{4}$$

It is easy to see that required distribution functions of the random variables (3) can be expressed through distribution function (4) as

$$F_Y(H) = P[Y < H] = F_2(H, H, X_2), \tag{5}$$

$$F_x(X) = P[x_m < X] = \int_{-\infty}^{\infty} \left[\left.\frac{\partial F_2(u,v,X)}{\partial u}\right|_{v=u}\right] du\,. \tag{6}$$

For joint probability density of magnitude and location of the greatest maximum of random process $y(x)$ the following expression is valid

$$W(u,X) = \frac{\partial}{\partial X}\left[\left.\frac{\partial F_2(u,v,X)}{\partial u}\right]\right|_{v=u}. \tag{7}$$

While integrating probability density $W(u,X)$ on the first argument within infinite limits, it is possible to find the probability density of the greatest maximum location of Markov random process $y(x)$:

$$W_x(X) = \int_{-\infty}^{\infty} W(u,X)\,du = \frac{\partial}{\partial X}\int_{-\infty}^{\infty}\left[\left.\frac{\partial F_2(u,v,X)}{\partial u}\right]\right|_{v=u} du\,. \tag{8}$$

(obviously, similar result can be received by differentiation of expression (6) on a variable X). Therefore, for calculation of required distributions of magnitude and location of the greatest maximum of Markov random process the function (4) should be found. For this purpose, we introduce auxiliary random process

$$z(x)=\begin{cases} u-y(x), & X_1 \le x \le X\,, \\ v-y(x), & X < x \le X_2\,, \end{cases} \tag{9}$$

and rewrite Eq. (4) in the form of

$$F_2(u,v,X)=P\left[\sup_{X_1\le x\le X_2} z(x)>0\right]. \tag{10}$$

According to Eqs. (1), (2), (9), the process $z(x)$ is Markov random process with drift K_{1z} and diffusion K_{2z} coefficients of a kind

$$K_1=\begin{cases} -a_1\,, & X_1 \le x \le x_0, \\ a_2\,, & x_0 < x \le X_2\,, \end{cases} \qquad K_{2z}=\begin{cases} b_1\,, & X_1 \le x \le x_0\,, \\ b_2\,, & x_0 < x \le X_2\,. \end{cases} \tag{11}$$

Then for probability (10) it can be written down [7]

$$F_2(u,v,X)=\int_0^\infty W(z,X_2)\,dz\,. \tag{12}$$

where $W(z,x)$ is one-dimensional probability density of random process $z(x)$ realizations which have never reached the borders $z=0$ and $z=\infty$ in the interval $[X_1,x]$.

As process $z(x)$ being Markov, the function $W(z,x)$ can be found from the solution of the direct Fokker-Planck-Kolmogorov (FPK) equation [7, 8]:

$$\frac{\partial W(z,x)}{\partial x}-\frac{\partial}{\partial z}\left[K_{1z}W(z,x)\right]-\frac{1}{2}\frac{\partial^2}{\partial z^2}\left[K_{2z}W(z,x)\right]=0 \tag{13}$$

with starting condition

$$W(z,x)=W(z,X_1) \tag{14}$$

and boundary conditions

$$W(0,x)=W(\infty,x)=0\,. \tag{15}$$

Here $W(z,X_1)$ is probability density of the random variable $z(X_1)=u-y(X_1)$. We designate $W_0(y)$ as probability density of the random variable $y(X_1)$. Then,

$$W(z,X_1)=W_0(u-z). \tag{16}$$

As drift and diffusion coefficients (11) discontinue jump in point $x = x_0$, we will try solution of equation (13) for $X \le x_0$ and $X > x_0$ separately. Let at first $X \le x_0$. We find solutions of Eq. (13) over segments $[X_1, X]$, $[X, x_0]$, $[x_0, X_2]$ serially. For this purpose, we make the change of variables

$$W(z,x) = U(z,x)\exp\left(\frac{K_{1z}}{K_{2z}}z - \frac{K_{1z}^2}{2K_{2z}}x\right), \tag{17}$$

$$U(z,x) = W(z,x)\exp\left(-\frac{K_{1z}}{K_{2z}}z + \frac{K_{1z}^2}{2K_{2z}}x\right), \tag{18}$$

which, if drift and diffusion coefficients are constant, reduces equation (13) to a canonical form for introduced function $U(z,x)$ (18) [7, 8]

$$\frac{\partial U(z,x)}{\partial x} = \frac{K_{2z}}{2}\frac{\partial^2 U(z,x)}{\partial z^2}. \tag{19}$$

According to Eqs. (14), (15), (18), we write down starting and boundary conditions for Eq. (19) as

$$U(z,X_1) = W(z,X_1)\exp\left(-\frac{K_{1z}}{K_{2z}}z + \frac{K_{1z}^2}{2K_{2z}}X_1\right), \tag{20}$$

$$U(0,x) = U(\infty,x) = 0. \tag{21}$$

Equation-solving procedures (19) are considered in detail in works [7, 8]. Using results [7, 8], taking into account Eqs. (20), (21) we receive the following expression for function $U(z,x)$:

$$U(z,x) = \frac{1}{\sqrt{2\pi K_{2z}(x - X_1)}}\int_0^\infty U(\xi, X_1)\chi[z,\xi,K_{2z}(x - X_1)]\,d\xi, \tag{22}$$

where it is designated

$$\chi(x_1,x_2,x_3) = \exp\left[-\frac{(x_1 - x_2)^2}{2x_3}\right] - \exp\left[-\frac{(x_1 + x_2)^2}{2x_3}\right].$$

Substituting Eq. (20) in Eq. (22) and, then, Eq. (22) in Eq. (17) we find the solution of FPK equation in the interval $x \in [X_1, X]$:

$$W(z,x) = \frac{1}{\sqrt{2\pi K_{2z}(x - X_1)}} \exp\left[-\frac{K_{1z}^2 (x - X_1)}{2K_{2z}}\right] \times$$
$$\times \int_0^\infty W(\xi, X_1) \exp\left[-\frac{K_{1z}}{K_{2z}}(\xi - z)\right] \chi[z, \xi, K_{2z}(x - X_1)] d\xi . \tag{23}$$

Considering Eq. (11), we rewrite last expression in a form of

$$W(z,x) = \frac{\exp\left[-a_1^2 (x - X_1)/2b_1\right]}{\sqrt{2\pi b_1 (x - X_1)}} \int_0^\infty W(\xi, X_1) \exp\left[-\frac{a_1}{b_1}(\xi - z)\right] \chi[z, \xi, b_1 (x - X_1)] d\xi . \tag{24}$$

Under $x = X$ we use Eq. (24) as starting condition $W(z, X)$ for solution of equation (13) over the interval $x \in [X, x_0]$. According to Eq. (9) under $x = X$ the random process $z(x)$ undergoes a jump by value $v - u$. Therefore, in order to satisfy to continuity property of probability density $W(z,x)$, the following equality should be carry out

$$W(z, X + 0) = W(z + u - v, X - 0). \tag{25}$$

Then, taking into account Eqs. (24), (25) the expression for $W(z, X)$ assumes the form

$$W(z, X) = \frac{1}{\sqrt{2\pi b_1 (X - X_1)}} \exp\left[-\frac{a_1^2 (X - X_1)}{2b_1}\right] \times$$
$$\times \int_0^\infty W(\xi, X_1) \exp\left[-\frac{a_1}{b_1}(\xi - z + v - u)\right] \chi[z + u - v, \xi, b_1 (X - X_1)] d\xi . \tag{26}$$

By means of change of variables (18), we finalize FPK equation (13) to a canonical form (19) and write down its solution similarly Eq. (22):

$$U(z,x) = \frac{1}{\sqrt{2\pi b_1 (x - X)}} \int_0^\infty W(\xi_1, X) \exp\left(\frac{a_1}{b_1}\xi_1 + \frac{a_1^2}{2b_1} X\right) \chi[z, \xi_1, b_1 (x - X_1)] d\xi_1 . \tag{27}$$

Substituting Eq. (26) in Eq. (27) and returning to probability density $W(z,x)$ in accordance with Eq. (17), we receive the solution of FPK equation (13) in the interval $x \in [X, x_0]$:

$$W(z,x) = \frac{1}{2\pi b_1 \sqrt{(x-X)(X-X_1)}} \exp\left[-\frac{a_1^2}{2b_1}(x-X_1) - \frac{a_1}{b_1}(z+u-v)\right] \times$$
$$\times \int_0^\infty \int_0^\infty W(\xi, X_1) \exp\left(\frac{a_1}{b_1}\xi\right) \chi[\xi_1 + u - v, \xi, b_1(X - X_1)] \chi[z, \xi_1, b_1(x - X)] d\xi d\xi_1 . \tag{28}$$

Further, using Eq. (28) in the point $x = x_0$ as starting condition for solution of equation (13), (19) over the interval $[x_0, X_2]$, similarly Eq. (27) and taking into account change of drift and diffusion coefficients (11), we find expression for function $U(z,x)$ in a form of

$$U(z,x) = \frac{1}{\sqrt{2\pi b_2 (x - x_0)}} \int_0^\infty W(\xi_2, x_0) \exp\left(-\frac{a_2}{b_2}\xi_2 + \frac{a_2^2}{2b_2} x_0\right) \chi[z, \xi_2, b_2(x - x_0)] d\xi_2 . \tag{29}$$

Now substituting Eq. (28) in Eq. (29) under $x = x_0$ and, then, Eq. (29) in Eq. (17), we have for probability density $W(z,x)$ in the segment $x \in [x_0, X_2]$

$$W(z,x) = \frac{\exp\left[-a_1(u-v)/b_1 + a_2 z/b_2 - a_1^2(x_0 - X_1)/2b_1 - a_2^2(x - x_0)/2b_2\right]}{2\pi b_1 \sqrt{2\pi b_2 (x - x_0)(x_0 - X)(X - X_1)}} \times$$
$$\times \int_0^\infty \int_0^\infty \int_0^\infty W(\xi, X_1) \exp\left[\frac{a_1}{b_1}\xi - \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)\xi_2\right] \chi[\xi_1 + u - v, \xi, b_1(X - X_1)] \times \tag{30}$$
$$\times \chi[\xi_2, \xi_1, b_1(x_0 - X)] \chi[z, \xi_2, b_2(x - x_0)] d\xi d\xi_1 d\xi_2 .$$

In an analogous way we find the solution of equation (13) over intervals $[X_1, x_0]$, $[x_0, X]$, $[X, X_2]$ serially under $X > x_0$. It is easy to see that the solution in the first segment $[X_1, x_0]$ coincides with Eq. (24). Assuming in Eq. (24) $x = x_0$, we receive the starting condition for probability density $W(z,x)$ (13) in the segment $[x_0, X]$. By means of change of variables (18), we lead FPK

equation (13) to a canonical form (19) and write down its solution taking into account Eq. (20) likewise Eq. (22):

$$U(z,x)=\frac{1}{\sqrt{2\pi b_2(x-x_0)}}\int_0^\infty W(\xi_1,x_0)\exp\left(-\frac{a_2}{b_2}\xi_1+\frac{a_2^2}{2b_2}x_0\right)\chi[z,\xi_1,b_2(x-x_0)]d\xi_1 . \tag{31}$$

Substituting Eq. (24) in Eq. (31) under $x=x_0$ and returning with help of Eq. (17) to probability density $W(z,x)$, we receive the solution of FPK equation (13) in the segment $x\in[x_0,X]$:

$$W(z,x)=\frac{\exp\left[-a_1^2(x_0-X_1)/2b_1-a_2^2(x-x_0)/2b_2\right]}{2\pi\sqrt{b_1b_2(x_0-X_1)(x-x_0)}}\int_0^\infty\int_0^\infty W(\xi,X_1)\times$$
$$\times\exp\left[\frac{a_1}{b_1}\xi-\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)\xi_1+\frac{a_2}{b_2}z\right]\chi[\xi_1,\xi,b_1(x_0-X_1)]\chi[z,\xi_1,b_2(x-x_0)]d\xi d\xi_1 . \tag{32}$$

Under $x=X$ we use Eq. (32) as the starting condition for the solution of equation (13) over the interval $x\in[X,X_2]$ taking into account discontinuous jump of the random process $z(x)$ by value $v-u$ in the point $x=X$, i.e. the fulfillment of the condition (25). Then, similarly Eq. (31), under $x\in[X,X_2]$ the solution of equation (19) can be written down in a kind of

$$U(z,x)=\frac{1}{\sqrt{2\pi b_2(x-X)}}\int_0^\infty W(\xi_2+u-v,X)\times$$
$$\times\exp\left(-a_2\xi_2/b_2+a_2^2X/2b_2\right)\chi[z,\xi_2,b_2(x-X)]d\xi_2 , \tag{33}$$

from which substituting Eq. (32) in Eq. (33) and Eq. (33) in Eq. (17) we find for probability density $W(z,x)$

$$W(z,x)=\frac{\exp\left[a_2(z+u-v)/b_2-a_1^2(x_0-X_1)/2b_1-a_2^2(x-x_0)/2b_2\right]}{2\pi b_1\sqrt{2\pi b_2(x-x_0)(x_0-X)(X-X_1)}}\times$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty W(\xi,X_1)\exp\left[\frac{a_1}{b_1}\xi-\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)\xi_1\right]\chi[\xi_1,\xi,b_1(x_0-X_1)]\times \tag{34}$$
$$\times\chi[\xi_2+u-v,\xi_1,b_2(X-x_0)]\chi[z,\xi_2,b_2(x-X)]d\xi d\xi_1 d\xi_2 , \quad x\in[X,X_2].$$

Now having substituted the serially found solutions (30) and (34) of FPK equation (13) for cases $X \le x_0$, $X > x_0$ in expression (12), we receive required distribution function (4) after simple transformations as follows

$$F_2(u,v,X) = \frac{\exp\left[-a_1(u-v)/b_1 - a_1^2(x_0 - X_1)/2b_1\right]}{2\pi b_1\sqrt{(x_0 - X)(X - X_1)}} \times$$
$$\times \int_0^\infty\int_0^\infty\int_0^\infty W_0(u-\xi)\exp\left[-\frac{a_1}{b_1}(\xi_2 - \xi)\right]\varphi\left(\frac{2a_2}{\sqrt{b_2}}, X_2 - x_0, \frac{2a_2\xi_2}{b_2}\right)\times \tag{35}$$
$$\times\chi[\xi_1 + u - v, \xi, b_1(X - X_1)]\chi[\xi_2, \xi_1, b_1(x_0 - X)]\,d\xi d\xi_1 d\xi_2\,, \qquad X \le x_0\,,$$

and

$$F_2(u,v,X) = \frac{\exp\left[-a_2(u-v)/b_2 - a_1^2(x_0 - X_1)/2b_1\right]}{2\pi\sqrt{b_1 b_2(X - x_0)(x_0 - X_1)}}\int_0^\infty\int_0^\infty\int_0^\infty W_0(u-\xi)\times$$
$$\times\exp\left[\frac{a_1}{b_1}\xi + \frac{a_2}{b_2}\xi_2 - \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)\xi_1 - \frac{a_2^2}{2b_2}(X - x_0)\right]\varphi\left(\frac{2a_2}{\sqrt{b_2}}, X_2 - X, \frac{2a_2\xi_2}{b_2}\right)\times \tag{36}$$
$$\times\chi[\xi_2 + u - v, \xi_1, b_2(X_2 - x_0)]\chi[\xi_1, \xi, b_1(x_0 - X_1)]\,d\xi d\xi_1 d\xi_2\,, \qquad X > x_0\,.$$

Here it is designated

$$\varphi(y_1, y_2, y_3) = \Phi\left(y_1\sqrt{y_2}/2 + y_3/y_1\sqrt{y_2}\right) - \exp(-y_3)\Phi\left(y_1\sqrt{y_2}/2 - y_3/y_1\sqrt{y_2}\right),$$

and $\Phi(x) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{x}\exp\left(-t^2/2\right)dt$ is probability integral [9]. Assuming in Eq. (36) $u = v = H$, $X = X_2$, we find distribution function (5) of magnitude of the greatest maximum of Markov random process

$$F_Y(H) = \frac{\exp\left[-a_1^2(x_0 - X_1)/2b_1\right]}{\sqrt{2\pi b_1(x_0 - X_1)}}\int_0^\infty\int_0^\infty W_0(u-\xi)\exp\left[\frac{a_1}{b_1}(\xi - \xi_1)\right]\times \tag{37}$$
$$\times\varphi\left(2a_2/\sqrt{b_2}, X_2 - x_0, 2a_2\xi_1/b_2\right)\chi[\xi_1, \xi, b_1(x_0 - X_1)]\,d\xi d\xi_1\,.$$

In a particular case, when the initial probability density (14) is described by Gaussian distribution law with mathematical expectation m_{X1} and dispersion σ_{X1}^2, i.e.

$$W_0(y)=\exp\left[(y-m_{X1})^2/2\sigma_{X1}^2\right]/\sigma_{X1}\sqrt{2\pi}\,, \tag{38}$$

we receive under substitution Eq. (38) in Eq. (37) and carrying out integration operation on a variable ξ

$$F_Y(H)=\frac{\exp\left[-a_1^2(x_0-X_1)/2\left(b_1(x_0-X_1)+\sigma_{X1}^2\right)\right]}{\sqrt{2\pi\left[b_1(x_0-X_1)+\sigma_{X1}^2\right]}}\int_0^\infty \exp\left[\frac{a_1(x_0-X_1)(H-m_{X1}-\xi)}{b_1}\right]\times$$
$$\times\left\{\exp\left[\frac{(H-m_{X1}-\xi)^2}{2\left(b_1(x_0-X_1)+\sigma_{X1}^2\right)}\right]\Phi\left[\frac{a_1\sigma_{X1}^2(x_0-X_1)+b_1(x_0-X_1)(H-m_{X1})+\xi\sigma_{X1}^2}{\sigma_{X1}\sqrt{b_1(x_0-X_1)\left(b_1(x_0-X_1)+\sigma_{X1}^2\right)}}\right]-\right.$$
$$\left.-\exp\left[-\frac{4a_1\xi\sigma_{X1}^2+b_1(H-m_{X1}+\xi)^2}{2\left(b_1(x_0-X_1)+\sigma_{X1}^2\right)}\right]\Phi\left[\frac{a_1\sigma_{X1}^2(x_0-X_1)+b_1(x_0-X_1)(H-m_{X1})-\xi\sigma_{X1}^2}{\sigma_{X1}\sqrt{b_1(x_0-X_1)\left(b_1(x_0-X_1)+\sigma_{X1}^2\right)}}\right]\right\}\times$$
$$\times\left\{\Phi\left(a_2\sqrt{\frac{X_2-x_0}{b_2}}+\frac{\xi}{\sqrt{b_2(X_2-x_0)}}\right)-\exp\left(-\frac{2a_2\xi}{b_2}\right)\Phi\left(a_2\sqrt{\frac{X_2-x_0}{b_2}}-\frac{\xi}{\sqrt{b_2(X_2-x_0)}}\right)\right\}d\xi\,. \tag{39}$$

Substituting distribution function (35), (36) in the formula (7), we find joint probability density of magnitude and location of the greatest maximum of Markov random process under $X\le x_0$:

$$W(u,X)=\frac{\exp\left[-a_1^2(x_0-X_1)/2b_1\right]}{\pi b_1^2(X-X_1)^{3/2}(x_0-X)^{3/2}}\int_0^\infty\int_0^\infty \xi\xi_1 W_0(u-\xi)\varphi\left(\frac{2a_2}{\sqrt{b_2}},X_2-x_0,\frac{2a_2\xi_1}{b_2}\right)\times$$
$$\times\exp\left[-\frac{\xi^2}{2b_1(X-X_1)}-\frac{\xi_1^2}{2b_1(x_0-X)}-\frac{a_1}{b_1}(\xi_1-\xi)\right]d\xi d\xi_1\,, \tag{40}$$

and under $X>x_0$

$$W(u,X)=\frac{1}{\pi(X-x_0)^{3/2}\sqrt{b_2(x_0-X)}}\left\{\frac{\exp\left[-a_2^2(X_2-X)/2b_2\right]}{\sqrt{2\pi b_2(X_2-X)}}+\frac{a_2}{b_2}\Phi\left[\sqrt{\frac{a_2^2}{b_2}(X_2-X)}\right]\right\}\times$$
$$\times\int_0^\infty\int_0^\infty \xi_1 W_0(u-\xi)\exp\left[\frac{a_1}{b_1}\xi-\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)\xi_1-\frac{\xi_1^2}{2b_1(X-x_0)}\right]\chi\left[\xi,\xi_1,b_1(x_0-X_1)\right]d\xi d\xi_1\,. \tag{41}$$

Using expressions (40), (41) in Eq. (8), we arrive to probability density of location of the greatest maximum of Markov random process of the form

$$W(X)=\frac{1}{\Delta}\begin{cases} z_1^2\Psi\left(z_1^2\dfrac{x_0-X}{\Delta},z_1^2\dfrac{x_0-X_1}{\Delta},z_2^2\dfrac{X_2-x_0}{\Delta},\dfrac{1}{R}\right), & X\le x_0\,, \\ z_2^2\Psi\left(z_2^2\dfrac{X-x_0}{\Delta},z_2^2\dfrac{X_2-x_0}{\Delta},z_2^2\dfrac{x_0-X_1}{\Delta},R\right), & X>x_0\,, \end{cases} \tag{42}$$

where it is designated

$$z_1^2=2a_1^2\Delta/b_1\,,\qquad z_2^2=2a_2^2\Delta/b_2\,,\qquad R=a_1b_2/a_2b_1\,, \tag{43}$$

$$\Psi(y,y_1,y_2,y_3)=\frac{1}{2\sqrt{\pi}|y|^{3/2}}\left\{\frac{\exp\left[-(y_1-y)/4\right]}{\sqrt{\pi(y_1-y)}}+\Phi\left(\sqrt{\frac{y_1-y}{2}}\right)\right\}\times$$
$$\times\int_0^\infty \xi\exp\left[-\frac{(\xi+y)^2}{4y}\right]\left[\Phi\left(\frac{y_3\xi+y_2}{\sqrt{2y_2}}\right)-\exp(-y_3\xi)\Phi\left(\frac{-y_3\xi+y_2}{\sqrt{2y_2}}\right)\right]d\xi\,.$$

3 Asymptotic Characteristics of Location of the Greatest Maximum of Markov Random Process with Piecewise Constant Drift and Diffusion Coefficients

Let us consider characteristics of location of the greatest maximum of Markov random process in the limiting case. As it is known [1, etc], the most important statistical characteristics of a random variable are its first two moments. In this connection we find a mathematical expectation $\langle X_m\rangle$ and moment of the second order $\langle X_m^2\rangle$ for deviation $X_m=x_m-x_0$ of location of the greatest maximum from value x_0:

$$\langle X_m\rangle=\int_{X_1}^{X_2}(x-x_0)W(x)\,dx=\frac{\Delta}{z_2^2}F_b\left(z_2^2\frac{X_2-x_0}{\Delta},z_1^2\frac{x_0-X_1}{\Delta},R\right)-$$
$$-\frac{\Delta}{z_1^2}F_b\left(z_1^2\frac{x_0-X_1}{\Delta},z_2^2\frac{X_2-x_0}{\Delta},\frac{1}{R}\right), \tag{44}$$

$$\left\langle X_m^2 \right\rangle = \int_{X_1}^{X_2} (x - x_0)^2 W(x)\, dx = \frac{\Delta^2}{z_2^4} F_V\left(z_2^2 \frac{X_2 - x_0}{\Delta}, z_1^2 \frac{x_0 - X_1}{\Delta}, R \right) +$$

$$+ \frac{\Delta^2}{z_1^4} F_V\left(z_1^2 \frac{x_0 - X_1}{\Delta}, z_2^2 \frac{X_2 - x_0}{\Delta}, \frac{1}{R} \right).$$

Here

$$F_b(y_1, y_2, y_3) = \int_0^{y_1} y\, \Psi(y, y_1, y_2, y_3)\, dy = \int_0^{y_1} \frac{1}{2\sqrt{\pi y}} \left\{ \frac{\exp[-(y_1 - y)/4]}{\sqrt{\pi(y_1 - y)}} + \Phi\left(\sqrt{\frac{y_1 - y}{2}} \right) \right\} \times$$

$$\times \int_0^{\infty} \xi \exp\left[-\frac{(\xi + y)^2}{4y} \right] \left[\Phi\left(\frac{y_3 \xi + y_2}{\sqrt{2y_2}} \right) - \exp(-y_3 \xi) \Phi\left(\frac{-y_3 \xi + y_2}{\sqrt{2y_2}} \right) \right] d\xi dy\,,$$

$$F_V(y_1, y_2, y_3) = \int_0^{y_1} y^2\, \Psi(y, y_1, y_2, y_3)\, dy = \frac{1}{2} \int_0^{y_1} \sqrt{\frac{y}{\pi}} \left\{ \frac{\exp[-(y_1 - y)/4]}{\sqrt{\pi(y_1 - y)}} + \Phi\left(\sqrt{\frac{y_1 - y}{2}} \right) \right\} \times$$

$$\times \int_0^{\infty} \xi \exp\left[-\frac{(\xi + y)^2}{4y} \right] \left[\Phi\left(\frac{y_3 \xi + y_2}{\sqrt{2y_2}} \right) - \exp(-y_3 \xi) \Phi\left(\frac{-y_3 \xi + y_2}{\sqrt{2y_2}} \right) \right] d\xi dy\,.$$

In order to investigate the asymptotic behavior of probability density (42) and moments (44), we carry out change of variables:

$$\mu_m = \begin{cases} z_1^2 X_m / \Delta\,, & X_m \le 0\,, \\ z_2^2 X_m / \Delta\,, & X_m > 0\,. \end{cases} \tag{45}$$

Probability density of the random variable μ_m has the appearance

$$W(\mu) = \begin{cases} \Psi\left[-\mu, z_1^2 (x_0 - X_1)/\Delta, z_2^2 (X_2 - x_0)/\Delta, 1/R \right], & \mu \le 0\,, \\ \Psi\left[\mu, z_2^2 (X_2 - x_0)/\Delta, z_1^2 (x_0 - X_1)/\Delta, R \right], & \mu > 0\,. \end{cases} \tag{46}$$

As it is known [7, 8], the drift coefficient characterizes rate of change of a regular component of a random process, and diffusion coefficient – a random component power. Therefore, it is possible to consider that dimensionless sizes z_i characterize a signal-to-noise ratio. Under $a_i >> b_i$ that is equivalent $z_i >> 1$, $i = 1,2$, the second and third arguments of function (43) in Eq. (46) can be approached infinity, so

$$\Psi(x,\infty,\infty,y)=W_a(x,y)=\Phi\left(\sqrt{|x|/2}\right)-1+ \\ +(2y+1)\exp\left[|x|y(y+1)\right]\left[1-\Phi\left((2y+1)\sqrt{|x|/2}\right)\right]. \tag{47}$$

Using representation (47) we write down the asymptotic expression for probability density (46) as

$$W(\mu)=\begin{cases} W_a(-\mu,1/R), & \mu\le 0, \\ W_a(\mu,R), & \mu>0. \end{cases} \tag{48}$$

Under $R=1$, that corresponds to the case $K_1=a\,\mathrm{sgn}(x_0-x)$, $K_2=b$, where $\mathrm{sgn}(x)=-1$ if $x\le 0$, and $\mathrm{sgn}(x)=1$ if $x>0$, the distribution (47) assumes the form

$$W_a(x,1)=\Phi\left(\sqrt{|x|/2}\right)-1+3\exp(2|x|)\left[1-\Phi\left(3\sqrt{|x|/2}\right)\right], \tag{49}$$

This probability density coincides with found in [10]. Its properties are in detail studied in [1, 3, 11]. So, the distribution (49) ordinary moments of odd order are equal to zero: $\left\langle \mu_m^{2k-1}\right\rangle=0$, $k=1,2,\ldots$, and for the ordinary moments of even order the following relationship holds [11]

$$\left\langle \mu_m^{2k}\right\rangle=2^{2k+1}\left\{\frac{1}{3^{4k+1}}\frac{d^{2k}}{d\alpha^{2k}}\left[\frac{1-\sqrt{1-2\alpha}}{\alpha\sqrt{1-2\alpha}}\right]_{\alpha=4/9}-\frac{(4k+1)!!}{2k+1}\right\},\quad k=1,2,\ldots \tag{50}$$

From here, in particular, follows that the random variables described by probability density (49) have a zero mathematical expectation and coefficient of skewness, dispersion equal to 13/2, and coefficient of excess equal to $1779/169\approx 10.53$.

Under $R\ne 1$ for ordinary moment of n-th order of the random variable μ_m we find

$$\left\langle \mu_m^{n}\right\rangle=\mu_n(R)+(-1)^n\mu_n(1/R),$$

$$\mu_n(y)=2^n\left\{\frac{1}{(2y+1)^{2n+1}}\frac{d^n}{d\alpha^n}\left[\frac{1-\sqrt{1-2\alpha}}{\alpha\sqrt{1-2\alpha}}\right]_{\alpha=2y(y+1)/(2y+1)^2}-\frac{(2n+1)!!}{n+1}\right\}.$$

In particular, mathematical expectation and second ordinary moment of the random variable μ_m described by distribution (48) have the appearance

$$\langle \mu_m \rangle = (R-1)/(R+1), \qquad \left\langle \mu_m^2 \right\rangle = 4 + 10R/(R+1)^2 .$$

Taking into account Eqs. (45), (46), it is possible to write down the asymptotic expressions for a mathematical expectation and mean-square deviation (44) location of the greatest maximum x_m of random process $y(x)$ from point x_0:

$$\langle X_m \rangle = \Delta\left[z_1^2 R(R+2) - z_2^2(2R+1)\right] / z_1^2 z_2^2 (R+1)^2 ,$$
$$\left\langle X_m^2 \right\rangle = 2\Delta^2 \left[z_1^4 R\left(2R^2 + 6R + 5\right) + z_2^4\left(5R^2 + 6R + 2\right)\right] / z_1^4 z_2^4 (R+1)^3 .$$

Generally, the ordinary moment of n-th order of the random variable X_m decreases proportionally to $z_{1,2}^{-2n}$.

4 Distribution Function of the First-Passage Time by Markov Random Process with Piecewise Constant Drift and Diffusion Coefficients

Now let us find the distribution function of the first-passage time x' for bound h by Markov random process $y(x)$. Distribution function of the random variable x' is by definition [7, 8]

$$F(h, X) = P[x' < X] = 1 - \tilde{F}(h, X), \tag{51}$$

where

$$\tilde{F}(h, X) = P\left[\sup_{X_1 \le x \le X} y(x) < h\right]$$

is probability of bound h non-crossing by process $y(x)$ over the interval $[X_1, X]$. Using auxiliary random process $z(x) = h - y(x)$, similarly Eq. (12), we can write down

$$\tilde{F}(h, X) = P\{ z(x) > 0, x \in [X_1, X] \} = \int_0^\infty W(z, X)\, dz , \tag{52}$$

where $W(z,x)$ is one-dimensional probability density of random process $z(x)$ realizations which have never reached the borders $z=0$ and $z=\infty$ in the interval $[X_1, X]$.

Function $W(z,x)$ is solution of FPK equation (13) [7, 8]. We find function (51) for $X \le x_0$ and $X > x_0$ separately. Under $X \le x_0$ let us use the solution (24) of FPK equation substituting which in the formula (52), and, then, Eq. (52) in Eq. (51), we find

$$F(h,X)=1-\frac{\exp\left[-a_1^2(X-X_1)/2b_1\right]}{\sqrt{2\pi b_1(X-X_1)}}\int_0^\infty\int_0^\infty W(\xi,X_1)\exp\left[\frac{a_1}{b_1}(\xi-z)\right]\chi[z,\xi,b_1(X-X_1)]d\xi dz.$$

After integral calculation on a variable *z*, with the account Eq. (16), we receive final expression for distribution function of the first-passage time, if $X \le x_0$:

$$F(h,X)=1-\int_0^\infty W_0(h-\xi)\left\{\Phi\left(\frac{\sqrt{X-X_1}}{2}+\frac{\xi}{\sqrt{X-X_1}}\right)-\exp(-\xi)\Phi\left(\frac{\sqrt{X-X_1}}{2}-\frac{\xi}{\sqrt{X-X_1}}\right)\right\}d\xi.$$

Under $X > x_0$ we use the solution (32) of FPK equation, substituting which in Eq. (52), and, then, Eq. (52) in Eq. (51) we have

$$F(h,X)=1-\frac{\exp\left[-a_1^2(x_0-X_1)/2b_1\right]}{2\pi\sqrt{b_1b_2(x_0-X_1)(X-x_0)}}\int_0^\infty\int_0^\infty\int_0^\infty W(\xi,X_1)\chi[\xi_1,\xi,b_1(x_0-X_1)]\times$$
$$\times\chi[z,\xi_1,b_1(X-x_0)]\exp\left[\frac{a_1}{b_1}\xi-\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)\xi_1+\frac{a_2}{b_2}z-\frac{a_2^2}{2b_2}(X-x_0)\right]d\xi d\xi_1 dz.$$

After integral calculation on a variable *z*, with the account Eq. (16), we receive final expression for distribution function of the first-passage time, if $X > x_0$:

$$F(h,X)=1-\frac{\exp\left[-a_1^2(x_0-X_1)/2b_1\right]}{\sqrt{2\pi b_1(x_0-X_1)}}\int_0^\infty\int_0^\infty W_0(h-\xi)\exp\left[\frac{a_1}{b_1}(\xi-\xi_1)\right]\times$$
$$\times\chi[\xi_1,\xi,b_1(x_0-X_1)]\varphi\left[2a_2/\sqrt{b_2},X-x_0,2a_2\xi_1/b_2\right]d\xi d\xi_1.$$

Some particular cases of the results stated here for definition of operating effectiveness of concrete detectors and measurers of the discontinuous quasi-deterministic and random signals can be found in [1, 2, 4]. In [5, 6] applicability is shown of the formulas (39), (50) for the statistical analysis of detection and measuring algorithms of abrupt changes in Gaussian random processes.

5 Conclusion

For definition of limiting characteristics of Markov random processes with piecewise constant drift and diffusion coefficients, the approach can be effectively used, based on the introduction of two-dimensional distribution function of the greatest maximum of random process, expressed through probability density, which is defined from the Fokker-Planck-Kolmogorov equation. Solving the specified equation in intervals of a constancy of drift and diffusion coefficients consistently, it is possible to find the closed analytical expressions for the distribution functions of magnitude, location and first-passage time of Markov random process. These formulas presuppose relatively simple approximations in the conditions of great signal-to-noise ratios.

The received results are applicable for the analysis of optimal and quasi-optimal processing algorithms of discontinuous processes where the solving statistics possesses Markovian properties.

Acknowledgements. The reported study was supported by Russian Science Foundation (research project No. 14-49-00079).

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Received: August 10 2014; Published: October 22, 2014