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$(2, 3, t)$-Generations for the Suzuki's Sporadic Simple Group Suz

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Abstract

A group G is called $(2, 3, t)$-generated if it can be generated by an element x of order 2 and an element y of order 3 such that the product xy has order t. In the present article we determine all the $(2, 3, t)$-generations for the Suzuki's sporadic simple group Suz, where t is an odd divisor of $|\mathrm{Suz}|$. This extends the earlier results of Mehrabadi, Ashrafi and Iranmanesh [9].

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1 Introduction and Preliminaries

Group generation has played and continued to play a significant role in solving outstanding problems in diverse areas of mathematics such as topology, geometry and number theory. A group G is said to be $(2, 3, t)$-generated if it can be generated by just two of its elements x and y such that $o(x) = 2$, $o(y) = 3$ and $o(xy) = t$. In this case, G is a factor of the modular group $PSL_2(\mathbf{Z})$, which is free product of two groups of order two and three. An important type of $(2, 3, t)$-generated groups are when $t = 7$. Such groups are called Hurwitz groups. Recently, there has been considerable amount of interest in the determination of $(2, 3, t)$-generations of the simple groups. Moori in [10] determined all possible $(2,3,p)$-generations for the Fischer group Fi_{22}. Ganief and Moori [8] computed $(2,3,t)$-generations of the Janko group J_3. More recently, in a series of articles [1], [2], [3], [4] and [5], the author with others investigated $(2,3,t)$-generations for the sporadic simple groups He, Co_1, Co_2, Co_3 and Ru. In the present article, we investigate $(2,3,t)$-generations for the Suzuki's sporadic simple group Suz where t is an odd divisor of |Suz|.

Throughout this paper our notation are standard and taken from [3]. In particular, let G be a finite group, A, B and C are classes of conjugate elements of G and if z is a fixed representative of C then $\xi_G(A,B,C)$ denotes the structure constant of the group algebra $Z(\mathbf{C}[\mathbf{G}])$, which is equal to the number of ordered pairs (x,y) such that $x \in A$, $y \in B$ and $xy = z$. It is well known that the number $\xi_G(A,B,C)$ can be calculated by the formula $\xi_G(A,B,C) = \frac{|A||B|}{|C|}\sum_{i=1}^{k}\frac{\chi_i(x)\chi_i(y)\overline{\chi_i(z)}}{\chi_i(1)}$ where $\chi_1, \chi_2, ..., \chi_k$ are irreducible complex characters of G. Further, let $\xi_G^*(A,B,C)$ denotes the number of distict ordered pairs (x,y) with $x \in A$, $y \in B$, $xy = z$ and $G = \langle x,y\rangle$. If there exists conjugacy classes A, B and C such that $\xi_G^*(A,B,C) > 0$, then we say that the group G is $(A,B,C)-$generated and (A,B,C) is called a generating triple for G. If H is a subgroup of G containing z and K is a conjugacy class of H such that $z \in K$, then $\sigma_H(A,B,K)$ denotes the number of distinct pairs (x,y) such that $x \in A$, $y \in B$, $xy = z$ and $\langle x,y\rangle \leq H$.

We compute the values of $\xi_G(A,B,C)$ and $\sigma_G(A,B,C)$ with the aid of computer algebra system **GAP** [11]. The **ATLAS** [6] is a valuable source of information and we will use its notation for conjugacy classes, maximal subgroups, character tables, permutation characters, etc. A general conjugacy class of elements of order n in G is denoted by nX. For examples, $2B$ represents the second conjugacy class of involutions in a group G. The number of conjugates of a given subgroup H of a group G containing the fixed element z is given by $h = \chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is a permutation character of G with action on the conjugates of H. In most cases, we will compute this value by using the conjugacy classes of $N_G(H)$ and the fusion map of $N_G(H)$ into G in the following theorem.

Theorem 1.1 *([8]) Let G be a finite group and H a subgroup of G containing a fixed element x such that $gcd(o(x),[N_G(H){:}H]) = 1$. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m}\frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|} \quad ,$$

where $x_1, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class $[x]_G$.

Lemma 1.2 *([7]) Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\xi^*(G) = \xi^*_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then $\xi^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

2 Main Results

The Suzuki group Suz is a sporadic simple group of order 448345497600 = $2^{13}.3^7.5^2.7.11.13$. The existence of Suzuki group Suz was first discovered by M. Suzuki. Later, Leech in 1965 rediscovered the group Suz using the Leech lattice. It is well known that Suz has exactly 43 conjugacy classes of its elements and 17 conjugacy classes of its maximal subgroups as listed in the **ATLAS** [6]. It has precisely two classes of involutions and three classes of elements of order 3, namely $2A$, $2B$, $3A, 3B$ and $3C$ as represented in the ATLAS. In this section we investigate all possible $(2, 3, t)$-generations for the Suzuki group Suz, where t is any odd divisor of $|\mathrm{Suz}|$. If the group Suz is $(2, 3, t)$-generated then it is well known that $\frac{1}{2} + \frac{1}{3} + \frac{1}{t} < 1$. Thus we only need to consider the cases that $t = 7, 9, 11, 13, 15, 21$. Since the cases when t is prime has already been discussed in [9], so it is enough to investigate the cases $t = 9, 15, 21$. Throughout this section we assume that $X \in \{A, B\}$ and $Y \in \{A, B, C\}$.

Lemma 2.1 *The Suzuki group Suz is $(2X, 3Y, 9Z)$-generated for $Z \in \{A, B\}$, if and only if $(X, Y) = (B, C)$.*

Proof: The Suzuki group Suz has two classes of elements of order 9 denoted by $9A$ and $9B$. Thus for $(2X, 3Y, 9Z)$-generations of the Suzuki group Suz we need to investigate the following twelve cases.

Case $(2X, 3A, 9Z)$. For triples in this cases, non-generation follows immediately since $\xi_{\mathrm{Suz}}(2X, 3A, 9Z) = 0$. Thus the group Suz is not $(2A, 3A, 9Z)$-, and $(2B, 3A, 9Z)$-generated.

Case $(2A, 3D, 9Z)$ *where* $D \in \{B, C\}$. In this case we compute the algebra constants as $\xi_{\mathrm{Suz}}(2A, 3B, 9Z) = 18$ and $\xi_{\mathrm{Suz}}(2A, 3C, 9Z) = 36$. Since $|C_{\mathrm{Suz}}(7A)| = 54$, we obtain $\xi_{\mathrm{Suz}}(2A, 3D, 9Z) < |C_{\mathrm{Suz}}(9Z)|$. Now by an application of Lemma 2, we conclude that $\xi^*_{\mathrm{Suz}}(2A, 3D, 9Z) = 0$ and hence the group Suz is not of type $(2A, 3B, 9A)$, $(2A, 3B, 9B)$, $(2A, 3C, 9A)$ and $(2A, 3C, 9B)$.

Case $(2B, 3B, 9Z)$. The only maximal subgroups of the Suzuki group Suz with orders divisible by 9 and having non-empty intersection with the classes $2B$, $3B$ and $9Z$, up to isomorphism, are $3_2.U_4(3).2_3$, $2^{1+6}.U_4(2)$ and $3^{2+4}{:}2(2^2 \times A_4)2$. By considering the fusions from these maximal subgroups to the group Suz we obtain $\sigma_{3_2.U_4(3).2_3}(2B, 3B, 9Z) = 0 = \sigma_{3^{2+4}:2(2^2 \times A_4)2}(2B, 3B, 9Z)$. Therefore, $2^{1+6}.U_4(2)$ is the only maximal subgroup that may contain $(2B, 3B, 9Z)$-generated subgroup. Let z be a fixed element of order 9 in the group Suz. Now an application of Theorem 1 shows that z is contained in precisely three conjugates of $2^{1+6}.U_4(2)$ and we calculate $\xi^*_{\mathrm{Suz}}(2B, 3B, 9Z) = \xi_{\mathrm{Suz}}(2B, 3B, 9Z) - 3 \times \sigma_{2^{1+6}.U_4(2)}(2B, 3B, 9Z) = 81 - 3(27) = 0$. Hence the Suzuki group Suz is not $(2B, 3B, 9A)$-, and $(2B, 3B, 9B)$-generated.

Case $(2B, 3C, 9Z)$. Direct computation in **GAP** using the character table of Suz shows that the structure constant $\xi_{\mathrm{Suz}}(2B, 3C, 9Z) = 648$. The only maximal subgroups of Suz with elements of order 9 and having non-trivial intersection with classes $2B$, $3C$ and $9Z$, up to isomorphism, are $3_2.U_4(3).2_3$ and $3^{2+4}{:}2(2^2 \times A_4)2$. An easy computation reveals that $\sigma_{3_2.U_4(3).2_3}(2B, 3C, 9Z) = 0 = \sigma_{3^{2+4}:2(2^2\times A_4)2}(2B, 3C, 9Z)$. Thus Suz has no proper $(2B, 3C, 9Z)$-generated subgroup and it follows that $\xi^*_{\mathrm{Suz}}(2B, 3C, 9Z) = \xi_{\mathrm{Suz}}(2B, 3C, 9Z) = 648$. Hence the Suzuki group Suz is $(2B, 3C, 9A)$-, and $(2B, 3C, 9B)$-generated. This completes the proof. □

Lemma 2.2 *The group Suz is $(2X, 3Y, 15Z)$-generated for $Z \in \{A, B, C\}$, if and only if $(X, Y, Z) \in \{(B, B, A), (B, B, B), (B, C, A), (B, C, B), (B, C, C)\}$.*

Proof: Set $\mathcal{S} = \{(A, A, A), (A, A, B), (A, A, C), (B, A, A), (B, A, B), (B, A, C), (A, B, A), (A, B, B)\}$. For $(X, Y, Z) \in \mathcal{S}$ we compute the algebra constants and in each case we obtain $\xi_{\mathrm{Suz}}(2X, 3Y, 15Z) = 0$. Therefore, $\xi^*_{\mathrm{Suz}}(2X, 3Y, 15Z) = 0$ for $(X, Y, Z) \in \mathcal{S}$ and non-generation of triples in this case follows.

For the triple $(2A, 3B, 15C)$ we calculate $\xi_{\mathrm{Suz}}(2A, 3B, 15C) = 10$ and $|C_{\mathrm{Suz}}(15C)| = 15$. Thus by Lemma 2, the group Suz is not $(2A, 3B, 15C)$-generated.

Next we consider the triples $(2A, 3C, 15A)$ and $(2A, 3C, 15B)$. By looking at the fusion map from maximal subgroups into the group Suz we see that $3^{2+4}{:}2(A_4 \times 2^2).2$ is the only maximal subgroup of Suz that may contains $(2A, 3C, 15A)$-, and $(2A, 3C, 15B)$-generated proper subgroups. If z is a fixed element of order 15 in group Suz then z is contained in precisely three conjugates copies of $3^{2+4}{:}2(A_4 \times 2^2).2$. Further since $\sigma_{\mathrm{Suz}}(2A, 3C, 15A) = 15 = \sigma_{\mathrm{Suz}}(2A, 3C, 15B)$. and we have $\xi^*_{\mathrm{Suz}}(2A, 3C, 15AB) = \xi_{\mathrm{Suz}}(2A, 3C, 15AB) - 3\sigma_{\mathrm{Suz}}(2A, 3C, 15AB) = 45 - 3(15) = 0$ where $15AB$ denotes the class $15A$ or $15B$. Thus the Suzuki group Suz is not of type $(2A, 3C, 15A)$ and $(2A, 3C, 15B)$. Similarly for the triple $(2B, 3B, 15C)$ we show that $\xi^*_{\mathrm{Suz}}(2B, 3B, 15C) = \xi_{\mathrm{Suz}}(2B, 3B, 15C) - 4\sigma_{2^{4+6}:3A_6}(2B, 3B, 15C) = 60 - 4(15) = 0$, and non-generation of the group Suz by the triple $(2B, 3B, 15C)$ follows.

Now, we consider the triple $(2B, 3B, 15X)$. The only maximal subgroup of the group Suz with order divisible and having non-empty intersection with classes $2B$, $3B$, $15A$ and $15B$ of Suz is isomorphic to $(3^2{:}4 \times A_6).2$ but our computation shows that $\sigma_{\mathrm{Suz}}(2B, 3B, 15X) = 0$. Hence, $\xi^*_{\mathrm{Suz}}(2B, 3B, 15X) = \xi_{\mathrm{Suz}}(2B, 3B, 15X) = 45$, showing that $(2B, 3B, 15A)$ and $(2B, 3B, 15B)$ are not generating triples of the group Suz.

Next, we consider the remaining triples $(2A, 3C, 15C)$, $(2B, 3C, 15X)$ and $(2B, 3C, 15C)$. For these triples we use "standard generators" of the group Suz given by Wilson in [12]. The group Suz has a 142-dimensional irreducible representation over $\mathbf{GF(2)}$. We generate the group Suz by using this representation, Suz $= \langle a, b\rangle$ where a and b are 142×142 matrices over $\mathbf{GF(2)}$ with orders 2 and 3 respectively such that ab has order 13. We see that $a \in 2B$, $b \in 3B$. We produce $c = ((ab)^6(ba)^2b^2ab^2(ab)^4(ba)^3b^2abab^2)^{12} \in 2A$, $d = ((ab)^5(ba)^2b^2ab^2(ab)^4(ba)^3b^2aba^3)^5 \in 3C$ such that $cd \in 15C$. Let $H = \langle c, d\rangle$ then H is a subgroup of Suz. We compute that $\sigma_H(2A, 3C, 15C) = 20$ and z is contained in exactly four conjugates of H. Thus, $\xi^*_{\text{Suz}}(2A, 3C, 15C) = \xi_{\text{Suz}}(2A, 3C, 15C) - 4\sigma_H(2A, 3C, 15C) = 90 - 4(20) < |C_{\text{Suz}}(15C)|$, and we have $\xi^*_{\text{Suz}}(2A, 3C, 15C) = 0$ proving that Suz is not $(2A, 3C, 15C)$-generated. Further we produce $e = a^b$, $f = e^b$, $g = (ab)^3(ba)^2b^2$, $h = (dg)^5$ then $e \in 2B$, $f \in 2B$, $g \in 8C$, $h \in 3C$ and $fh \in 15A$. let $K = \langle f, h\rangle$ then $|K| = 251596800$ and $K \cong G_2(4)$. By investigating the maximal subgroups of $G_2(4)$ and the fusion map of $G_2(4)$ into Suz we obtain $\xi^*_{\text{Suz}}(2B, 3C, 15A) = \xi_{\text{Suz}}(2B, 3C, 15A) - 3\sigma_K(2B, 3C, 15A) < |C_{\text{Suz}}(15X)|$. Similar results also holds for the triple $(2B, 3C, 15B)$. Hence the group Suz is not $(2B, 3C, 15A)-$, and $(2B, 3C, 15B)-$generated.

Finally in the case of triple $(2B, 3C, 15C)$, we have $\xi_{\text{Suz}}(2B, 3C, 15C) = 1035$. Again by using the above discussed standard generators for sporadic simple group Suz we produce $l = ((ab)^5(ba)^2b^2ab^2(ab)^4(ba)^3b^2aba^3)^{40} \in 3C$ such that $e \in 2B$, $l \in 3C$ and $el \in 15C$. Let M be a subgroup generated by e and l then we show that $M \leq$ Suz and there exists elements of order 5, 7, 11 and 13. Since Suz contains no proper subgroup with order divisible by $5 \times 7 \times 11 \times 13$, we have $M =$ Suz and therefore Suz is $(2B, 3C, 15C)$-generated. This completes the proof. □

Lemma 2.3 *The Suzuki group Suz is* $(2X, 3Y, 21Z)$*-generated, where* $Z \in \{A, B\}$*, if and only if* $(X, Y) \in \{(A, C), (B, B), (B, C)\}$.

Proof: The conjugacy class $(21B)^{-1} = 21A$ and results obtained by replacing one of the classes $21A$, $21B$ by the other are the same. Let $21Z$ denotes the class $23A$ or $23B$.

For the triples $(2A, 3A, 21Z)$ and $(2B, 3A, 21Z)$ non-generation of the group Suz follows immediately since $\xi_{\text{Suz}}(2A, 3A, 21Z) = 0 = \xi_{\text{Suz}}(2B, 3A, 21Z)$.

For the triple $(2B, 3B, 21Z)$, the only maximal subgroup of the group Suz with order divisible 21 that may contains $(2B, 3B, 21Z)$-generated proper subgroups is isomorphic to $3_2.U_4(3).2_3$. However $\sigma_{3_2.U_4(3).2_3}(2B, 3B, 21Z) = 0$ and this shows that there is no contribution from the maximal subgroup $3_2.U_4(3).2_3$ towards the structure constant $\xi_{\text{Suz}}(2B, 3B, 21Z)$. Hence $\xi^*_{\text{Suz}}(2B, 3B, 21Z) = \xi_{\text{Suz}}(2B, 3B, 21Z) = 56$, proving that Suz is $(2B, 3B, 21Z)$-generated.

Next, we consider the triple $(2B, 3C, 21Z)$. The maximal subgroups of Suz that have non-empty intersection with the Suz-claases $2B$, $3C$ and $21Z$ are, up to isomorphism, $G_2(4)$, $3_2.U_4(3).2_3$ and $A_4 \times PSL(3,4)$. We calculate that $\xi_{\text{Suz}}(2B, 3C, 21Z) = 819$, $\sigma_{G_2(4)}(2B, 3C, 21Z) = 336$, $\sigma_{3_2.U_4(3).2_3}(2B, 3C, 21Z) = 0$ and $\sigma_{A_4\times PSL(3,4)}(2B, 3C, 21Z) = 63$. Further, a fixed element of order 21 in Suz is contained in a unique conjugate of each of $G_2(4)$ and $A_4 \times PSL(3,4)$. We obtain $\xi^*_{\text{Suz}}(2B, 3C, 21Z) \geq \xi_{\text{Suz}}(2B, 3C, 21Z) - \sigma_{G_2(4)}(2B, 3C, 21Z) - \sigma_{A_4\times PSL(3,4)}(2B, 3C, 21Z) = 819 - 336 - 63 = 420$. Thus, Suz is $(2B, 3C, 21Z)$-generated.

Finally, we show that the group Suz is $(2A, 3C, 21Z)$-generated by using its standard generators as in the previous lemma. The group Suz $= \langle a, b\rangle$ such that $a \in 2B$, $b \in 3B$ and $o(ab) = 13$. By using generators a and b we produce $n = ((ab)^6(ba)^2b^2ab^2(ab)^4(ba)^3b^2abab^2)^{12}$, $p = ((ab)^5(ba)^2b^2ab^2(ab)^4(ba)^3b^2aba^3)^5$ $r = (np)^{10}$ then $n \in 2A$, $p \in 3C$, $r \in 2A$ and $rp \in 21A$.

Let Q be a subgroup generated by r and p then Q is a subgroup of Suz and there exists elements of order 5, 7, 11 and 13 in Q. Since no maximal subgroup of Suz has elements of these orders, we conclude that $Q =$ Suz proving that $(2A, 3C, 21Z)$ is a generating triple for the group Suz. This completes the proof. □

We now summarize our results in the following theorem:

Theorem 2.4 *Let Suz be the Suzuki's sporadic simple group. Then Suz is $(2, 3, t)$-generated, where t is an odd divisor of $|Suz|$ except $t = 7$.*

Proof: The result follows immediately from Lemmas 3, 4, 5, 6 and 7 together with results from [9] and [13].

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