Applied Mathematical Sciences, Vol. 8, 2014, no. 59, 2931 - 2940
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.44314

The Solution of Burger's Equation by Elzaki Homotopy Perturbation Method

Tarig M. Elzaki

King Abdulaziz University
Department of Mathematics
Jeddah 21589, Saudi Arabia

Hwajoon Kim*

Kyungdong University
School of IT Engineering
Goseong 219-705, Gangwon, Korea
* Corresponding author

Abstract

In this article, we have proposed a reliable combination of Elzaki transform and homotopy perturbation method(EHPM) to solve Burger's equations. The analytical results of Rurger's equations have been obtained in terms of convergent series with easily computable components, and the nonlinear terms in the equations can be handled by using of homotopy perturbation method(HPM). The results tell us that the proposed method is more efficient and easier to handle when is compared with existing other methods in such PDEs.

Mathematics Subject Classification: 35A08, 44A05

Keywords: Homotopy perturbation method, Burger's equations

1 Introduction

The nonlinear equations are the most important phenomena across the world. Nonlinear phenomena have important efficiency on applied mathematics, physics, and issues related to engineering. The importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics is still a big problem that needs new methods.

The concept of Elzaki transform[4-8, 12] was proposed by Tarig M. Elzaki, and the homoyopy perturbation method was proposed by He[11], who developed by combining the standard homotopy and perturbation method.

Burger's equation serves as useful model for many interesting problems in applied mathematics. It models effectively certain problems of a fluid flow nature, in which either shocks or viscous dissipation is significant factor. The first steady-state solutions of Burger's equation were given by Bateman[1] in 1915, and the equation got its name from the extensive research of Burger[2] beginning in 1939. Burger focused on modeling turbulence, but the equation is useful for modeling such diverse physical phenomena as shock flows, traffic flow, acoustic transmission in fog, etc.

In the resent years, many researchers mainly had paid attention to studying the solution of nonlinear partial differential equations by using various methods. Among these are the adomian decomposition method[10], homotopy perturbation method[9, 11, 16-18], variational iteration method, differential transform method[3, 18], and projected differential transform method[8].

In this article, we use Elzaki transform and homotopy perturbation method together to solve Burger's equations in (2+1)-dimensional, (3+1)-dimensional and (n+1)-dimensional with boundary conditions.

2 Elzaki Transforms and homotopy perturbation method

In this section, we review some basic facts for Elzaki Transforms and homotopy perturbation method.

2.1 Elzaki Transforms

The basic definition of the Elzaki transform[4-5] is defined as follows; The Elzaki transform of the function $f(t)$ is defined by

$$E\left[f(t)\right] = v\int_0^\infty f(t)\ e^{-\frac{t}{v}}dt \tag{1}$$

for $t > 0$.

Tarig M. Elzaki and Sailh M. Elzaki proposed Elzaki transform(or a modified Sumudu transform[13-14]) was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations in [6-9]. The transform is a novel tool for solving some differential equations which cannot be solved by Sumudu transform[5]. In this article, we have combined Elzaki transform and homotopy perturbation(EHPM) to solve Burger's equations.

Lemma 2.1 *Let $E[f(t)] = T(v)$. To obtain Elzaki transform of partial derivative we use integration by parts, and then we have[4]:*

$$\begin{aligned}
&a)\quad E[\frac{\partial f(x,t)}{\partial t}] = \frac{1}{v}T(x,v) - vf(x,0)\\
&b)\quad E[\frac{\partial^2 f(x,t)}{\partial t^2}] = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}\\
&c)\quad E[\frac{\partial f(x,t)}{\partial x}] = \frac{d}{dx}[T(x,v)]\\
&d)\quad E[\frac{\partial^2 f(x,t)}{\partial x^2}] = \frac{d^2}{dx^2}[T(x,v)]
\end{aligned}$$

We can easily extend this result to the n-th partial derivative by using the mathematical induction.

2.2 Homotopy Perturbation Method(HPM)

Let X and Y be the topological spaces. If f and g are continuous maps of the space X into Y, it is said that f is homotopic to g, if there is continuous map

$$F : X \times [0,1] \to Y$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each $x \in X$, then the map is called homotopy between f and g.

To explain the homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0 \tag{2}$$

for L is any differential operator. We define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u), \tag{3}$$

where $F(u)$ is a functional operator with known solution v_0 which can be obtained easily. It is clear that $H(u,0) = F(u)$ and $H(u,1) = L(u) = 0$, which are the linear and nonlinear original equations, respectively.

In topology, this show that $H(u,p)$ continuously traces an implicitly defined carves from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The HPM uses the embed ling parameter p as a small parameter for $0 \leq p \leq 1$, and write the solution as a power series

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots. \tag{4}$$

If $p \to 1$, then (4) corresponds to (3) and becomes the approximate solution of the form

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{5}$$

We assume that (5) has a unique solution. The comparisons of like powers of p give solutions of various orders.

3 The solution of Burger's Equation by Elzaki Homotopy Perturbation Method(EHPM)

For a given velocity u and viscosity coefficient ε, the general form of Burger's equation given by

$$\frac{\partial u}{\partial t} + (u.\nabla)u - \varepsilon \nabla^2 u = 0. \tag{6}$$

with the initial condition

$$u(x_1, x_2, ..., x_n, 0) = \sum_{i=1}^{n} x_i$$

where $u = u(x_1, x_2, ..., x_n, t), \nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$, and $\nabla^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Lemma 3.1 (Convolution theorem for Elzaki transform)

$$E(f * g) \ = \ \frac{1}{u} \ E(f)E(g)$$

for $E(f)$ is the Elzaki transform of f.

Theorem 3.2 *Let p is given by*

$$u(X, t) = \sum_{i=0}^{\infty} p^n u_n(X, t).$$

Then the solution of Burger's equation be expressed by

$$u(X, t) = \sum_{n=0}^{\infty} u_n(X, t)$$

where,

$$p^0 : u_0(X, t) = x_1 + x_2 + ... + x_n$$
$$p^n : u_n(X, t) = -E^{-1}\{v^2 E[H_{n-1}(u)]\}$$

for all natural number n.

Proof. Taking Elzaki transform on the equation (6), and making use of the differential property of Elzaki transform and the above initial condition, we have

$$E\left(u(X,t)\right) = v^2 \sum_{i=0}^{n} x_i - vE\left[(u(X,t).\nabla)u(X,t) - \varepsilon \nabla^2 u(X,t)\right], \tag{7}$$

where $X = x_1, x_2, ...x_n$.

Applying the inverse Elzaki transform of equation (7), we have

$$u(X,t) = \sum_{i=0}^{n} x_i - E^{-1}\left\{vE\left[(u(X,t).\nabla)u(X,t) - \varepsilon \nabla^2 u(X,t)\right]\right\}. \tag{8}$$

Applying the homotopy perturbation method

$$u(X,t) = \sum_{i=0}^{\infty} p^n u_n(X,t). \tag{9}$$

and by the lemma 3.1, we have

$$(u(X,t).\nabla)u(X,t) - \varepsilon \nabla^2 u(X,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{10}$$

where $H_n(u)$are He's polynomials given by

$$H_n(u_1, u_2, ..., u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[\sum_{i=0}^{\infty} p^i u_i\right], \tag{11}$$

for $n = 0, 1, 2, \cdots$. Substituting equations (9) and (10) into the equation (8), we get

$$\sum_{n=0}^{\infty} p^n u_n(X,t) = (x_1 + x_2 + ... + x_n) - p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right\}. \tag{12}$$

This is the coupling of Elzaki transform and the homotopy perturbation method (EHPM). Comparing the coefficient of like powers of p, the following approximations are obtained.

$$p^0 : u_0(X,t) = x_1 + x_2 + ... + x_n$$

$$p^n : u_n(X,t) = -E^{-1}\{v^2 E[H_{n-1}(u)]\}$$

for all natural number n. Then the solution is

$$u(X,t) = \sum_{n=0}^{\infty} u_n(X,t).$$

Example 3.4 Consider the (2+1)-dimensional Burger's equation

$$\frac{\partial u}{\partial t} + \left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} \right) - \varepsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \tag{14}$$

subject to the initial condition $u(x, y, 0) = x + y$.

Solution. Taking Elzaki transform on equation (14), we have

$$E(u(x, y, t)) = v^2(x + y) - vE\{(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y}) - \varepsilon(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})\}.$$

Next, let us take the inverse Elzaki transform to obtain

$$u(x, y, t) = (x + y) - E^{-1}\{vE[(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y}) - \varepsilon(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})]\}.$$

Now applying the homotopy perturbation method to get:

$$\sum_{n=0}^{\infty} p^n u_n(x, y, t) = x + y - p\{E^{-1}[vE[\sum_{n=0}^{\infty} p^n H_n(u)]]\}. \tag{15}$$

That is

$$\begin{aligned} p\left(uu_x + uu_y\right) - \varepsilon p\left(u_{xx} + u_{yy}\right) &= 0, \\ u = u_0 + pu_1 + p^2u_2 + \cdots&. \end{aligned} \tag{16}$$

The equation (16) can be written as

$$\begin{gathered} p[u_0 + pu_1 + p^2u_2 + ...][u_{0x} + pu_{1x} + p^2u_{2x} + \cdots] \\ +p[u_0 + pu_1 + p^2u_2 + \cdots][u_{0y} + pu_{1y} + p^2u_{2y} + \cdots] \\ -\varepsilon p[u_{0xx} + pu_{1xx} + p^2u_{2xx} + \cdots] \\ -\varepsilon p[u_{0yy} + pu_{1yy} + p^2u_{2yy} + \cdots] = 0. \end{gathered} \tag{17}$$

The first few components of He's polynomials, are given by

$$H_0(u) = u_0u_{0x} + u_0u_{0y} - \varepsilon u_{0xx} - \varepsilon u_{0yy}$$

$$H_1(u) = u_0u_{1x} + u_1u_{0x} + u_0u_{1y} + u_1u_{0y} - \varepsilon u_{1xx} - \varepsilon u_{1yy}$$

$$\cdots$$

Comparing the coefficients of the same powers of p, we get

$$\begin{aligned} p^0 : u_0(x, y, t) &= x + y, \ H_0(u) = 2(x + y) \\ p^1 : u_1(x, y, t) &= -E^{-1}\{vE[H_0(u)]\} = -2t(x + y), \\ H_1(u) &= -8t(x + y) \\ p^2 : u_2(x, y, t) &= -E^{-1}\{vE[H_1(u)]\} = 4t^2(x + y) \end{aligned}$$

$$\begin{array}{rcl} p^3 : u_3(x,y,t) &=& -8t^3(x+y) \\ p^4 : u_4(x,y,t) &=& 16t^4(x+y) \\ \cdots && \end{array}$$

Therefore the solution $u(x,y,t)$is given by

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \cdots = \frac{x+y}{1+2t}.$$

Example 3.5 Let us consider (3+1)-dimensional Burger's equation

$$\frac{\partial u}{\partial t} + \left(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial z}\right) - \varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0 \qquad (18)$$

subject to the initial condition $u(x,y,z,0) = x+y+z$.

Solution. By the same method in example 3.4, we applying homotopy perturbation method after taking Ezaki, and the inverse Elzaki transforms of the equation (18), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,y,t) = x+y+z - p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right\} \qquad (19)$$

where

$$\begin{aligned} H_0(u) &= u_0u_{0x} + u_0u_{0y} + u_0u_{0z} - \varepsilon(u_{0xx} + u_{0yy} + u_{0zz}) \\ H_1(u) &= u_0u_{1x} + u_1u_{0x} + u_0u_{1y} + u_1u_{0y} + u_0u_{1z} + u_1u_{0z} \\ &\quad -\varepsilon(u_{1xx} + u_{1yy} + u_{1zz}) \\ &\cdots \end{aligned}$$

Comparing the coefficients of the same powers of p, we get

$$\begin{array}{rcl} p^0 : u_0(x,y,z,t) &=& x+y+z, \\ H_0(u) &=& 3(x+y+z) \\ p^1 : u_1(x,y,z,t) &=& -E^{-1}\left\{vE\left[H_0(u)\right]\right\} = -3t(x+y+z), \\ H_1(u) &=& -18t(x+y+z) \\ p^2 : u_2(x,y,z,t) &=& -E^{-1}\left\{vE\left[H_1(u)\right]\right\} = 9t^2(x+y+z) \\ p^3 : u_3(x,y,z,t) &=& -27t^3(x+y+z) \\ \cdots && \end{array}$$

Then the solution $u(x,y,z,t)$ is given by

$$u(x,y,z,t) = u_0(x,y,z,t) + u_1(x,y,z,t) + u_2(x,y,z,t) + \cdots = \frac{x+y+z}{1+3t}.$$

Example 3.6 We would like to consider $(n+1)$-dimensional Burger's equation

$$\frac{\partial u}{\partial t}+\left(u\frac{\partial u}{\partial x_1}+u\frac{\partial u}{\partial x_2}+....+u\frac{\partial u}{\partial x_n}\right)$$
$$-\varepsilon\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+....+\frac{\partial^2 u}{\partial x_n^2}\right)=0 \qquad (19)$$

with the initial condition

$$u(x_1,x_2,\cdots,x_n,0)=x_1+x_2+\cdots+x_n.$$

Solution. By using the same method in the example 3.5, we find that

$$\sum_{n=0}^{\infty}p^n u_n(x_1,x_2,\cdots,x_n,t)$$
$$=(x_1+x_2+\cdots+x_n)-p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right\}$$

where

$$H_0(u)=\sum_{i=0}^{n}u_0u_{0x_i}-\varepsilon\sum_{i=0}^{n}u_{0x_ix_i},$$
$$H_1(u)=\sum_{i=0}^{n}(u_0u_{1x_i}+u_1u_{0x_i})-\varepsilon\sum_{i=0}^{n}u_{1x_ix_i}$$
$$\cdots$$

Comparing the coefficients of the same powers of p, we get

$$\begin{aligned}
p^0: u_0(X,t) &= \sum_{i=1}^{n}x_i,\ H_0(u)=n\sum_{i=1}^{n}x_i\\
p^1: u_1(X,t) &= -nt\sum_{i=1}^{n}x_i,\ H_1(u)=-2n^2t\sum_{i=1}^{n}x_i\\
p^2: u_2(X,t) &= n^2t^2\sum_{i=1}^{n}x_i\\
p^3: u_3(X,t) &= -n^3t^3\sum_{i=1}^{n}x_i\\
\cdots
\end{aligned}$$

where $X=x_1,x_2,x_3,\cdots,x_n$.

Consequently, we obtain

$$u(x_1,x_2,\cdots,x_n,t)=\sum_{i=0}^{n}x_i[1-nt+n^2t^2-n^3t^3+\cdots]$$
$$=\frac{\sum_{i=1}^{n}x_i}{1+nt}$$

which is an exact solution of equation (19).

The main concern of this paper was to combine Elzaki transform and homotopy perturbation method(EHPM). This method has been successfully employed to obtain the approximate solutions of Burger's equation. Also the obtained result by this method is almost accurate with the exact solution.

References

[1] Bateman, Some recent researches on the motion of fluids, *Mon. Wea. Rev.,* **43** (1915), 163-170.

[2] Burger, J. M., Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, *Trans. Roy. Neth. Acad. Sci.,* **17** (1939), 1-53.

[3] Sh. Chang, Il Chang, A new algorithm for calculating one-dimensional differential transform of nonlinear functions, *Appl. Math. & Compu.* **195** (2008), 799-808.

[4] T. M. Elzaki and S. M. Elzaki, Application of new transform 'Elzaki transform' to partial differential equations, *Glob. J. Pure & Appl. Math.* **1** (2011), 65-70.

[5] T. M. Elzaki, S. M. Elzaki, and E. M. A. Hilal, Elzaki and Sumudu transforms for solving some differential equations, *Glob. J. of Pure & Appl. Math.,* **8** (2012), 167-173.

[6] T. M. Elzaki, A solution for nonlinear systems of differential equations using a mixture of Elzaki transform and differential transform method, *Int. Math. For.,***7** (2012), 625-630.

[7] T. M. Elzaki and E. M. A. Hilal, Homotopy perturbation and Elzaki transform for solving nonlinear partial differential equations, *Math. Theo. & Model.,* **2** (2012), 33-42.

[8] T. M. Elzaki and E. M. A. Hilal, Solution of linear and nonlinear partial differential equations using mixture of Elzaki transform and the projected differential transform method, *Math. Theo. & Model.,* **2** (2012), 50-59.

[9] Francisco M. Fernandez, On the homotopy perturbation method for Boussinesq-like equations, *Appl. Math. & Compu.* **230** (2014), 208-210.

[10] Hashim, I, M. SM. Noorani, R. Ahmed, S. A. Bakar, E. S. Ismail and A. M. Zakaria, Accuracy of the adomian decomposition method applied to the Lorenz system, *chaos.* **08** (2006), 135-145.

[11] J. H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, *Int. J. Nonlin. Mech.* **35** (2008), 37-43.

[12] Hj. Kim, A Note on the Shifting Theorems for the Elzaki Transform, *Int. J. of Math. Anal.* **8** (2014), 491-488.

[13] A. Kilicman and H. E. Gadain, An application of double Laplace transform and Sumudu transform, *Loba. J. Math.* **30** (2009), 214-223.

[14] A. Kilicman and H. ELtayeb, A note on integral transform and partial differential equation, *Appl. Math. Sci.* **4** (2014), 109-118.

[15] P. R. Sharma and G. Methi, Applications of homotopy perturbation method to partial differential equations, *Asi. J. of Math. & Stat.* **4** (2011), 140-150.

[16] J. Singh and D. Sushila, Homotopy perturbation Sumudu transform method for nonlinear equations, *Adv. Theor. Appl. Mech.,* **4** (2011), 165-175.

[17] N. H. Sweilam and M.M. Khader, Exact solutions of some capled nonlinear partial differential equations using the homotopy perturbation method, *Compu.& Math. Appl.* **58** (2009), 2134-2141.

[18] N. Taghizadeh, M. Akbari and A. Ghelichzadeh, Exact solution of Burger's equations by homotopy perturbation method and reduced differential transformation method, *Aus. J. of Bas. & Appl. Sci.,* **5** (2011), 580-589.

Table. Elzaki transform of some Functions

$f(t)$	$E[f(t)] = T(v)$
1	v^2
t	v^3
t^n	$n!\ v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$sin\,at$	$\frac{av^3}{1+a^2v^2}$
$cos\,at$	$\frac{v^2}{1+a^2v^2}$

Received: April 28, 2014