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Very Simply Explicitly Invertible Approximations of Normal Cumulative and Normal Quantile Function

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Abstract

For the normal cumulative distribution function: Φ(x) we give the new approximation 2**(-22**(1-41**(x/10))) for any x>0, which is very simple (with only integer constants and operations - and / and power elevation **) and is very simply explicitly invertible having 1 entry of x. It has 3 decimals of precision having absolute error less than 0.00013. We compute the inverse which approximates the normal quantile function, or probit, and it has the relative precision of 1 percent (from 0.5) till beyond 0.999. We give an open problem and a noticeable bibliography. We report several other approximations.

Mathematics Subject Classification: 33B20 , 33F05 , 65D20 , 97N50

Keywords: normal distribution function, normal cumulative, normal cdf, Φ, normal quantile, probit, error function, erf, erfc, Q function, cPhi, inverse erf, erf^{-1}, approximation

1 Introduction

This paper deals with the approximation of 2 special functions, $\Phi(x)$ and ϕ_α. Let's remember that $\Phi(x)$ and its inverse $\phi_\alpha := \Phi^{-1}(\alpha)$ play a central role in Statistics, essentially as a consequence of the Central Limit Theorem.
Papers [11] and recent [24] list several approximations of $\Phi(x)$, which were published in literature directly as approximations, or bounds, for that function, or are immediately derived from approximations or bounds for related functions (see Remark 8 below), and give new ones.

Remark 1. Though computers now allow to compute them with arbitrary precision, such approximations are still valuable for several reasons, including *to catch the soul* of the considered functions, allowing to understand at a glance their behaviour. Furthermore, here we produce only *explicitly invertible* (and, in fact, *simply*) approximations, which allow to keep coherence working contemporarily with the considered functions and their respective inverses.
Let's add, finally, that despite technologic progress, those functions – of wide practical use – are not always available on pocket calculators.

Remark 2. The research about approximating $\Phi(x)$ floats among:

- exactness, but requiring limits, as series and continued fractions
- width of domains of approximation (usually $x \geq 0$ but not always)
- precision of approximations, but affecting their simplicity
- simplicity of approximations, but affecting their precision:
  - ⋄ there are *few* and/or *short* decimal constants
  - ⋄ if possible there are *no* decimal constants
- explicit invertibility by elementary functions.

Remark 3. The invertibility generates this categories:
(a) not explicitly invertible
(b) explicitly invertible solving a quartic equation
(c) explicitly invertible solving a *generic* cubic equation
(d) explicitly invertible solving a *particular* cubic equations $x^3 + ax + b = 0$
(e) *simply explicitly invertible* solving a quadratic (or biquadratic) equation
(f) *very simply explicitly invertible*, with only 1 entry of x.

Remark 4. Some special functions – among which those we consider in this paper – are monotonic and then invertible, though not by elementary functions.

Remark 5. Of course the inverse of an approximation of an invertible function f is an approximation (how good, it has to be seen) of the inverse of f.

Remark 6. Usually the approximations of $\Phi(x)$ are not designed to be explicitly invertible by means of elementary functions, but sometimes they are, solving cubic or quartic equations (after obvious substitutions) or rarely in simpler manners.

Remark 7. As well known, it is possible to explicitly solve cubic and even quartic equations, by complicate formulas, but it is *not* a standard procedure in usual mathematical practice. (In literature, such explicit invertibility usually is not even stated when presenting the approximations of $\Phi(x)$).

2 Preliminary Notes

Remark 8. Similar things as in Remarks 1-7 may be said for the functions erf(x), erfc(x) and $Q(x)$ we are going to define.

Definition 1. (Most standard; unluckily there are ambiguities in literature). Normal cumulative distribution function:

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \tag{1}$$

Error function:

$$erf(x) := \int_{0}^{x} \frac{2}{\sqrt{\pi}} e^{-t^2} dt \tag{2}$$

Q-function:

$$Q(x) := \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt. \tag{3}$$

Complementary error function:

$$erfc(x) = \frac{1}{2} + \int_{x}^{+\infty} \frac{2}{\sqrt{\pi}} e^{-t^2} dt\ . \tag{4}$$

Remark 8. Mutual relations, holding for any $x \in I\!R$:

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} erf\Big(\frac{x}{\sqrt{2}}\Big) \tag{5}$$

$$erf(x) = 2\Phi(x\sqrt{2}) - 1 \tag{6}$$

$$Q(x) := 1 - \Phi(x) \tag{7}$$

$$erfc(x) := 1 - erf(x) \tag{8}$$

Remark 9. We wrote := both in (3) and (7) because both are used as definitions in literature. We wrote := in (8) because that is usually used as definition, and not (4).

Remark 10. The approximation of $\Phi(x)$ for $x \geq 0$ and of its inverse for $0 \leq \alpha \leq \frac{1}{2}$ are sufficient because of symmetries:

$$\Phi(-x) = 1 - \Phi(x) \qquad \forall x \in I\!R \tag{9}$$

$$\phi_{1-\alpha} = -\phi_\alpha \qquad \forall \alpha \in]0,1[. \tag{10}$$

3 Our Results

3.1 New Approximation of $\Phi(x)$

Denoting by $|\varepsilon(x)|$ the absolute error and by $\varepsilon_r(x)$ the relative error, we give the following approximation:

$$(A) \quad \Phi(x) \simeq 2^{-22^{1-41^{x/10}}} \begin{cases} |\varepsilon(x)| < 1.28 \cdot 10^{-4} \\ \\ |\varepsilon_r(x)| < 1.66 \cdot 10^{-4} \end{cases} \qquad \forall x \geq 0$$

Let $\eta(x)$ be the approximation of $\Phi(x)$ considered in Formula (A):

$$\eta(x) := 2^{-22^{1-41^{x/10}}}.$$

The function $1.3 \cdot 10^{-4} - |\Phi(x) - \eta(x)|$ is positive for $0 \leq x \leq 5$ as may be seen by plotting it (see Figures 1 and 2). All the graphs may be obtained by professional software *Mathematica*$^{(R)}$ or for free at the site www.wolframalpha.com: for the considered[(1)] case, write Plot[
1.3 10ˆ(-4) - Abs[1/2+(1/2) Erf[x/Sqrt[2]] - 2ˆ(-22ˆ(1 - 41ˆ(x/10)))],{x,0,5}].

[1] All graphs may be obtained by these instructions, using as options (for example) WorkingPrecision -> 100, PlotStyle -> Black :

```
phi[x_] = 1/2+ (1/2) Erf[x/Sqrt[2]]
iphi[α_] = Sqrt[2] InverseErf[2 α - 1]
PHI41[x_] = 2^(−22^(1 − 41^(x/10)))
iPHI41[α_] = (10/Log[41]) Log[ 1 - (Log[(-Log[α])/Log[2]])/Log[22]]

Fig. 1 : Plot[{0, 128/10^6 - Abs[PHI41[x] - phi[x]]}, {x, 0, 5}, (options)]
Fig. 2 : Plot[{0, 128/10^6 - Abs[PHI41[x] - phi[x]]}, {x, 2.6, 2.8}...
Fig. 3 : Plot[{0, 166/10^6 - Abs[(PHI41[x] - phi[x])/phi[x]]}, {x, 0, 5}...
Fig. 4 : Plot[{0, 166/10^6 - Abs[(PHI41[x] - phi[x])/phi[x]]}, {x, 0.16, 0.18}...
Fig. 5 : Plot[{ 0, 5/1000 - Abs[iPHI41[x] - iphi[x]]} ,{ x, 0.5, 0.9926}...
Fig. 6 : Plot[{0, 5/1000 - Abs[iPHI41[x] - iphi[x]]}, {x, 0.9924, 0.9926}...
Fig. 7 : Plot[{0, 1/100 - Abs[(iPHI41[x] - iphi[x])/iphi[x]]}, {x, 0.5, 0.99909}...
Fig. 8 : Plot[{0, 1/100 - Abs[(iPHI41[x] - iphi[x])/iphi[x]]}, {x, 0.99907, 0.99909}...
```

For $x > 5$ let's consider that it is $\Phi(5) = 0.9999997...$ and $\Phi(x) \to 1$ and Φ is increasing, then

$$(\forall x > 5) \qquad |1 - \Phi(x)| < 10^{-6}. \tag{11}$$

It is, for $x > 5$,

$$x > 5 > 3.6378... = \frac{10}{\log 41} \log\Big(1 - \frac{1}{\log 22} \log\big(\frac{\log(1-10^{-4})}{-\log 2}\big)\Big)$$
$$10 \log_{41}(1 - \log_{22}(-\log_2(1 - 10^{-4}))) < x$$
$$\log_{41}(1 - \log_{22}(-\log_2(1 - 10^{-4}))) < x/10$$
$$1 - \log_{22}(-\log_2(1 - 10^{-4})) < 41^{x/10}$$
$$\log_{22}(-\log_2(1 - 10^{-4})) > 1 - 41^{x/10}$$
$$\log_2(1 - 10^{-4}) < -22^{1-41^{x/10}}$$
$$1 - 10^{-4} < 2^{-22^{1-41^{x/10}}}$$
$$0 < 1 - 2^{-22^{1-41^{x/10}}} < 10^{-4}$$

that is to say

$$(\forall x > 5) \qquad |1 - \eta(x)| < 10^{-4}. \tag{12}$$

By (11) and (12) it is

$$(\forall x > 5) \qquad |\Phi(x) - \eta(x)| \le |1 - \Phi(x)| + |1 - \eta(x)| < 10^{-6} + 10^{-4} < 1.3 \cdot 10^{-4}.$$

Then, for the relative error of Formula (A), for $0 \le x \le 5$, see Fig. 3 and Fig. 4, and for $x \ge 5$ it is $\Phi(x) > 0.9$ (see above) and then

$$\frac{|2^{-22^{1-41^{x/10}}} - \Phi(x)|}{|\Phi(x)|} < \frac{|2^{-22^{1-41^{x/10}}} - \Phi(x)|}{0.9} = \frac{|\varepsilon(x)|}{0.9} < \frac{1.3\,10^{-4}}{0.9} < 1.7\,10^{-4}.$$

Fig. 1 Absolute error [PHI41]

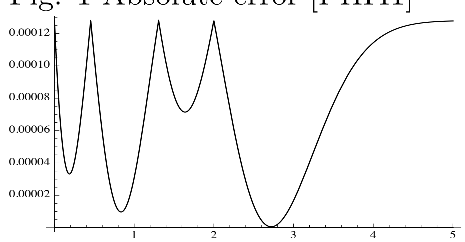


Fig. 2 Its zoom [PHI41Z]

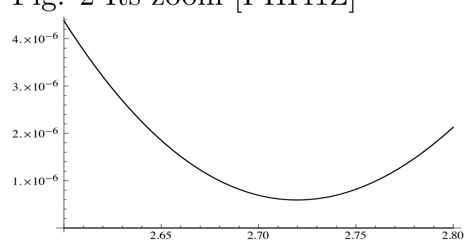


Fig. 3 Relative error [PHI41R]

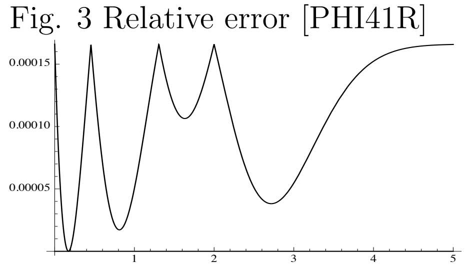
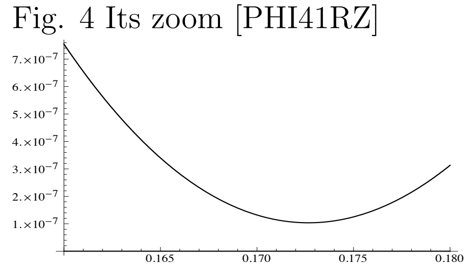


Fig. 4 Its zoom [PHI41RZ]



3.2 Inversion: Approximation of ϕ_α.

Remembering Remark 5, inverting (A), and still denoting by $|\varepsilon(x)|$ the absolute error and by $\varepsilon(x)$ the relative error, we give the following approximation of the normal quantile function $\phi_\alpha = \Phi^{-1}(\alpha)$:

$$(a) \quad \phi_\alpha \simeq \frac{10}{\log 41}\log\left(1-\frac{\log((-\log\alpha)/\log 2)}{\log 22}\right) \begin{cases} |\varepsilon(\alpha)| < 5\cdot 10^{-3} \ \forall\alpha\in[0.5, 9925] \\ |\varepsilon_r(\alpha)| < 1\% \ \forall\alpha\in[0.5, 0.99908] \end{cases}$$

For the absolute error of (a) see Figures 5 and 6. For the relative error of (a) see Figures 7 and 8.

Fig. 5 Absolute error [iPHI41]

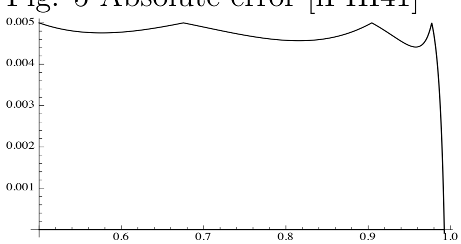


Fig. 6 Its zoom [iPHI41Z]

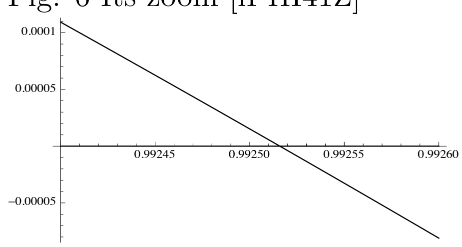


Fig. 7 Absolute error [iPHI41R]

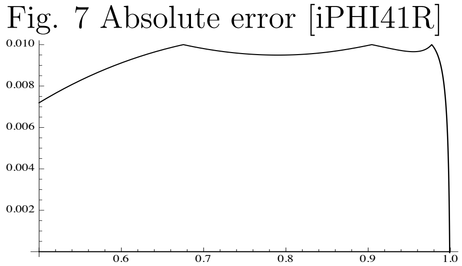
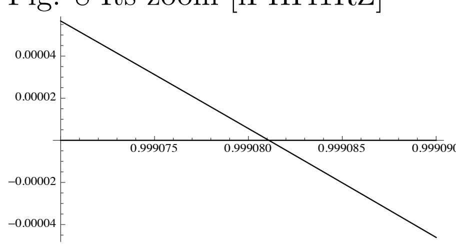


Fig. 8 Its zoom [iPHI41RZ]



4 Conclusions

In this paper for the normal cumulative distribution function $\Phi(x)$ and the normal quantile function ϕ_α respectively we gave these *very simply explicitly invertible* (with 1 entry of x) *corresponding* approximations:

$$(A) \quad \Phi(x) \simeq 2^{-22^{1-41^{x/10}}} \qquad \forall x \geq 0$$

$$(a) \quad \phi_\alpha \simeq \frac{10}{\log 41}\log\left(1-\frac{\log((-\log\alpha)/\log 2)}{\log 22}\right) \qquad 0.5 \leq \alpha < 1$$

As quantified more precisely in Sections 3.1 and 3.2, the approximation (A) of $\Phi(x)$ grants *abundantly* 3 decimals of precision (having absolute error less than 0.00013), is very simple – with only 1 entry of x – and very simply explicitly invertible, and the inverse (a) has essentiallly the same characteristics, giving an approximation of the normal quantile function ϕ_α which maintains the 1% precision (from 0.5) till 0.999???

In the end we remember that by the symmetry Formulas (9) and (10) the approximations of $\Phi(x)$ for $x \geq 0$ and of ϕ_α for $0.5 \leq \alpha < 1$ are sufficient.

Remark 11. Because of the mutual relations (see Remark 8) among the functions $\Phi(x)$, $\operatorname{erf}(x)$, $Q(x)$ and $\operatorname{erfc}(x)$, to approximate one of them is equivalent to approximate the others.
We searched in a wide literature approximations published not only for $\Phi(x)$, but also the approximations of $\Phi(x)$ implicitly contained in the approximations of the other 3 functions.

Remark 12. We will report other's Author's Formulas in a standard format. This allows easy comparison.
We use x as independent variable. We write $\Phi(x) \simeq$, and always consider both absolute and relative errors, in absolute value, and write respectively $|\varepsilon(x)|$ and $|\varepsilon_r(x)|$. Authors not always report both. And they write them with different precisions. We found and wrote those errors with 2 digits after decimal point, in the form $a.bc \cdot 10^{-n}$.
Of course any function may be written in several ways. We did our best in reporting other Author's formulas, sometimes changing the formal appearance. In particular

$$\frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{f(x)}} \quad = \quad 0.5 + 0.5(1 - \exp f(x))^{0.5} \quad = \quad \frac{1 + (1 - \exp f(x))^{\frac{1}{2}}}{2}$$

and we will write in the first way whenever possible.

Remark 13. The most recent approximation of $\Phi(x)$ we have found in literature is in paper [24] (2014), which gives this new approximation

$$\Phi(x) \simeq 1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x^2}{2}}}{0.226 + 0.64x + 0.33\sqrt{x^2 + 3}} \qquad x > 0$$

for which we found $|\varepsilon(x)| < 1.93 \cdot 10^{-4}$ and $|\varepsilon_r(x)| < 3.86 \cdot 10^{-4}$, not explicitly invertible. The same paper lists 16 other approximations of $\Phi(x)$; the last is

$$\Phi(x) \simeq \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\frac{x^2}{2}\frac{\frac{4}{\pi} + 0.147\frac{x^2}{2}}{1 + 0.147\frac{x^2}{2}}}} \tag{13}$$

holding for $x \geq 0$, for which we found $|\varepsilon(x)| < 6.21 \cdot 10^{-5}$ and $|\varepsilon_r(x)| < 6.30 \cdot 10^{-5}$, originally published in [109] as

$$erf(x) \simeq \sqrt{1 - e^{-x^2 \frac{\frac{4}{\pi}+0.147x^2}{1+0.147x^2}}} \qquad \forall x \geq 0\,. \tag{14}$$

Both (13) and (14) are explicitly invertible, essentially by solving a biquadratic equation, after obvious substitutions, just as the following improvements of (13) which we already made available on the net in [95]

$$\Phi(x) \simeq \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-x^2 \frac{17+x^2}{26.694+2x^2}}} \qquad \begin{cases} |\varepsilon(x)| < 4.00 \cdot 10^{-5} \\ \\ |\varepsilon_r(x)| < 4.53 \cdot 10^{-5} \end{cases} \qquad \forall x \geq 0$$

and in [94]

$$\Phi(x) \simeq \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{\frac{-1.2735457x^2 - 0.0743968x^4}{2+0.1480931x^2+0.0002580x^4}}} \qquad \begin{cases} |\varepsilon(x)| < 1.14 \cdot 10^{-5} \\ \\ |\varepsilon_r(x)| < 1.78 \cdot 10^{-5} \end{cases} \qquad \forall x \geq 0\,.$$

Both the above improvements reach 4 decimals of precision.

Remark 14. As far as we know, the most recent new approximations (all of 2013) of $Q(x)$ or $\mathrm{erf}(x)$ or $\mathrm{erfc}(x)$ (from which one could immediately obtain approximations of $\Phi(x)$) are this double inequality

$$\frac{1}{x + \sqrt{4+x^2}}\sqrt{\frac{2}{\pi}}\, e^{-\frac{x^2}{2}} \leq Q(x) \leq \frac{1}{\sqrt{x^2 + x + \frac{8}{\pi}}}\sqrt{\frac{2}{\pi}}\, e^{-\frac{x^2}{2}}$$

in [19] (originally published for $\sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}} Q(x)$, and notice that the lower bound is of [10]), this bound

$$Q(x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+x^2}} e^{-x^2/2}$$

in [39] (year 2013, originally published for $\sqrt{2\pi}Q(x)$) and a family

$$Q(x) \leq \Sigma_{k=0}^{n} \frac{a_k}{x}\, e^{-b_k x^2}$$

of upper bounds in [41] (year 2013 too) and this family

$$Q(x) \geq \Sigma_{k=0}^{n}\, a_k\, x\, e^{-b_k x^2}$$

of lower bounds in [42] (year 2013 too), and from those lower and upper bounds one *could* obtain approximations of $\Phi(x)$ which are not explicitly invertible by elementary functions. Those approximations are especiallly valuable not only because bounds, but also for little relative errors *for the function* $Q(x)$ for great values of x. (Notice that $Q(x) \to 0$).

Remark 15. As far as we know, the most recent new approximation of $\Phi(x)$ or $Q(x)$ or $\mathrm{erf}(x)$ or $\mathrm{erfc}(x)$, having 1 entry of x, is this

$$\Phi(x) \simeq 1 - 0.24015\, e^{-0.5616x^2}$$

originally published as

$$erfc(\sqrt{x}) \simeq \Sigma_{k=1}^{N} a_k e^{-kbx} \qquad N := 1 \quad a_1 = 0.4803; \quad b = 1.1232$$

in [78] and [79] (both year 2012); (then the Authors give other approximations, with 2 and 3 entries of x). Clearly that approximation is not intended to minimize the absolute error, which in 0 is about 0.52 for $\mathrm{erfc}(x)$ (and 0.26 for the derived approximation of $\Phi(x)$); and in fact its quality is the little relative error *for the function* $erfc(x)$ for great values of x. (Notice that $erfc(x) \to 0$).

Another recent (2009) approximation of $\Phi(x)$ (or $Q(x)$ or $\mathrm{erf}(x)$ or $\mathrm{erfc}(x)$) having 1 entry of x is this of [11]

$$\Phi(x) \simeq \frac{1}{1 + e^{-1.702x}} \qquad x \in \mathbb{R}$$

for which we found $|\varepsilon(x)| < 9.49 \cdot 10^{-3}$ and $|\varepsilon_r(x)| < 1.35 \cdot 10^{-2}$: it is simple and very simply explicitly invertible, but not so precise; the same paper gives also this approximation

$$\Phi(x) \simeq \frac{1}{1 + e^{-0.07056x^3 - 1.5976x}} \qquad x \in \mathbb{R}$$

for which we found $|\varepsilon(x)| < 1.42 \cdot 10^{-4}$ and $|\varepsilon_r(x)| < 2.08 \cdot 10^{-4}$, which is explicitly invertible solving a *particular* cubic equation.
Both the approximations have the quality of holding on the whole $\mathbb{R}$.

Remark 16. (Conclusions) As far as we know, before our Formula (A), the most precise (with respect both to the absolute error and to the relative error) approximation of $\Phi(x)$

(α) published as approximations or bounds for $\Phi(x)$ or $Q(x)$ or $\mathrm{erf}(x)$ or $\mathrm{erfc}(x)$
(β) holding at least for $x \geq 0$ (and, then, $\Phi(-x) = 1 - \Phi(x)$)
(γ) defined by a single expression (or, not piecewise defined)
(δ) very simply explicitly invertible, with 1 entry of x

was this of [6]

$$\Phi(x) \simeq \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{-\sqrt{\frac{\pi}{8}}x^2}} \qquad x \geq 0$$

for which we found $|\varepsilon(x)| < 1.98 \cdot 10^{-3}$ and $|\varepsilon_r(x)| < 2.04 \cdot 10^{-3}$. The Author provides also the inverse, approximating the normal quantile function ϕ_α.

Our Formula (A) approximating the normal cumulative distribution function $\Phi(x)$, having $|\varepsilon(x)| < 1.28 \cdot 10^{-4}$ and $|\varepsilon_r(x)| < 1.66 \cdot 10^{-4}$, appears really quite noticeable for simplicity, precision and explicit invertibility.
That makes also quite valuable our Formula (a) for the approximation of the normal quantile function ϕ_α inverse of $\Phi(x)$.

Remark 16. (Open problem). Modify constants to approximate $\mathrm{erf}(x)$ applying $\mathrm{erf}(x) = 2\Phi(x\sqrt{2}) - 1$ and our Formula (A), possibly avoiding $\sqrt{2}$.

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