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The Representation on Solutions of the Sine-Gordon and Klein-Gordon Equations by Laplace Transform

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Abstract

The Sine-Gordon equation is a nonlinear wave equation involving the d'Alembert operator and the sine of the unknown function, and it is involved in crystal dislocation. Intrinsically, this equation is closely connected Klein-Gordon equation. The Klein-Gordon equation is that of the motion of a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles. In this article, we have checked the representation on solutions of Sine-Gordon and Klein-Gordon equations by Laplace transform.

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1 Introduction

The Sine-Gordon equation was introduced by E. Bour in [1], and rediscovered by Frenkel and Kontorova in their research of crystal dislocation[5]. The equation is famous for the presence of soliton solutions. In the real space-time coordinates (x, t), the equation has a form of

$$u_{tt} - u_{xx} + sin\, u(x, t) = 0$$

for $u = u(x,t)$ is an unknown function. In the case of mechanical transmission line, $u(x,t)$ describes an angle of rotation of the pendulum, and this equation models many problems in quantum mechanics, solitons, and condensed physics[7]. If the function $sin\,u$ be changed to u, it is the famous Klein-Gordan equation[14, 16]. This equation is often abbreviated as the form of d'Alembert operator.

Several researches have been ongoing for many years with respect to the equation. The exact soliton solutions of Sine-Gordan equation have been obtained in [8], and Hesameddini[7] has employed Elzaki homotopy perturbation method[5-6] to obtain the approximate analytic solution of the Sine-Gordan and the Klein-Gordan equations. The local and global wellposedness of the Sine-Gordon equation in an appropriate function space proved by Pelinovsky and Sakovich[15], [9] has dealt with the finite volume method for solving the equation, Uddin has proposed a numerical method based on radial basis functions for solving the equation[19], and Zhao has presented a new hyperbolic auxiliary function method for obtaining traveling wave solutions of nonlinear partial differential equations[21].

In the latest papers, we have pursued the solution and its representation of several equations by integral transforms[2-3, 10-13, 17-18]. Here we have checked the solutions of Sine-Gordon and Klein-Gordon equations by Laplace transform when the initial conditions are constants, and the used tool is the method of variation of parameters. The main idea is that $\pounds[u(x,t)]$ can be considered as a function of x in solving process by Laplace transform, and this can be dealt with a linear equation. Of course, the used tool can be changed to another integral transform such as Elzaki[3-4, 10-11, 17-18] or Sumudu one[20].

2 The representation on solutions of the Sine-Gordon and Klein-Gordon equations by Laplace transform

Let us check the representation of Sine-Gordon and Klein-Gordon equations by Laplace transform.

Lemma 2.1

$$\pounds(sin\,u) = \frac{s\,sin\,u(x,0) + cos\,u(x,0)}{s^2+1}$$

for $u = u(t,x)$.

Proof. Trying a partial integration 2 times with respect to time variable t, we obtain the above result.

Lagrange's method gives a particular solution y_p of $y'' + p(x)y' + q(x)y = r(x)$ on open interval I in the form

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx \tag{1}$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous equation $y'' + p(x)y' + q(x)y = 0$ on I, and W is the Wronskian of y_1, y_2.

Theorem 2.2 *Let w be the unit step function. The solution $u(x,t)$ of Sine-Gordon equation*

$$u_{tt} - u_{xx} + sin\, u(x,t) = 0$$

can be represented by

$$u(x,t) = \ a(t+x)w(t+x) + b(t-x)w(t-x)$$

$$+ sin\alpha(1 - cos\, t) + cos\alpha(t - sin\, t) - (s\alpha + \beta)t$$

where $A(s) = \pounds[a(t)]$, $B(s) = \pounds[b(t)]$, $u(x,0) = \alpha$ and $u_t(x,0) = \beta$.

Proof. To begin with, let us put $U(x,s) = \pounds[u(x,t)]$, $u(x,0) = \alpha$ and $u_t(x,0) = \beta$. Taking Laplace transform with respect to t, we have

$$s^2 U - s\alpha - \beta - \frac{\partial^2 U}{\partial x^2} + \frac{s\, sin\, \alpha + cos\, \alpha}{s^2 + 1} = 0. \tag{2}$$

Organizing this equality, we have

$$\frac{\partial^2 U}{\partial x^2} - s^2 U = -s\alpha - \beta + \frac{s\, sin\, \alpha + cos\, \alpha}{s^2 + 1} = 0.$$

Since this equation is linear, and it may be regraded as ODE for $U(x,s)$ considered as a function of x. Implies, a general solution has the form of

$$U(x,s) = A(s)e^{sx} + B(s)e^{-sx} + U_p$$

for U_p is a particular solution of the equation (2). Now that let us find the U_p. The Wronskian W of e^{sx} and e^{-sx} is $W = 2s$, and

$$U_p = -\frac{1}{2s} e^{sx} \int e^{-sx} \Big(\frac{s\, sin\, \alpha + cos\, \alpha}{s^2 + 1} - s\alpha - \beta\Big)\, dx$$

$$+ \frac{1}{2s} e^{-sx} \int e^{sx} \Big(\frac{s\, sin\, \alpha + cos\, \alpha}{s^2 + 1} - s\alpha - \beta\Big)\, dx.$$

By the direct integration, we have

$$U_p = \frac{1}{s^2}(\frac{s\,sin\,\alpha + cos\,\alpha}{s^2+1} - s\alpha - \beta)\,dx.$$

Thus,

$$U(x,s) = A(s)e^{sx} + B(s)e^{-sx} + \frac{sin\,\alpha}{s(s^2+1)} + \frac{cos\,\alpha}{s^2(s^2+1)} - \frac{s\alpha+\beta}{s^2}$$

for $u(x,0) = \alpha$ and for $u_t(x,0) = \beta$. Taking the inverse Laplace transform, applying t-shifting theorem and scanning a table of Laplace transform, we have the solution

$$u(x,t) = a(t+x)w(t+x) + b(t-x)w(t-x)$$

$$+sin\alpha(1-cos\,t) + cos\alpha(t - sin\,t) - (s\alpha+\beta)t, \tag{3}$$

where $A(s) = \pounds[a(t)]$, $B(s) = \pounds[b(t)]$ and w is the unit step function.

Note that

$$\pounds[\frac{1}{s(s^2+w^2)}] = \frac{1}{w^2}(1-cos\,wt),$$

$$\pounds[\frac{1}{s^2(s^2+w^2)}] = \frac{1}{w^3}(wt - sin\,wt)$$

and $\pounds(t) = 1/s^2$.

Corollary 2.3 *The equation (3) can be rewritten as*

$$u(x,t) = a(t+x)w(t+x) + b(t-x)w(t-x)$$

$$+sin\,a(x)(1-cos\,t) + cos\,a(x)(t-sin\,t) - \frac{1}{2}(sa(x) + a'(x))t \tag{4}$$

for $x > 0$. Similarly, if $x < 0$, we have

$$u(x,t) = a(t+x)w(t+x) + b(t-x)w(t-x)$$

$$+sin\,b(-x)(1-cos\,t) + cos\,b(-x)(t-sin\,t) - \frac{1}{2}(sb(-x) + b'(-x))t \tag{5}$$

for w is the unit step function.

Proof. It is followed by the conditions of $u(x,0) = \alpha$ and $u_t(x,0) = \beta$.

Example 2.4

$$u_{tt} - u_{xx} + sin\,u = 0,\ \ u(x,0) = 0,\ \ u_t(x,0) = 0.$$

Solution. Taking Laplace transform with respect to t, we have

$$s^2U - \frac{\partial^2 U}{\partial x^2} + \frac{1}{s^2+1} = 0$$

for $U(x,s) = \pounds[u(x,t)]$. By the method similar to theorem 2, we have

$$U(x,s) = A(s)e^{sx} + B(s)e^{-sx} - \frac{1}{2s}e^{sx}\int e^{-sx}(\frac{1}{s^2+1})\,dx$$

$$+\frac{1}{2s}e^{-sx}\int e^{sx}(\frac{1}{s^2+1})\,dx.$$

Calculating the equality by the direct integration, we obtain

$$U(x,s) = A(s)e^{sx} + B(s)e^{-sx} + \frac{1}{s^2(s^2+1)}$$

and so,

$$u(x,t) = \, a(t+x)w(t+x) + b(t-x)w(t-x) + t - sin\,t,$$

where $A(s) = \pounds[a(t)]$, $B(s) = \pounds[b(t)]$ and w is the unit step function. Of course, this is the same result as the equation (3) of theorem 2. On the other hand, from the conditions of $u(x,0) = 0$ and $u_t(x,0) = 0$, we have the relation

$$a(x)w(x) + b(-x)w(-x) = 0$$

and

$$a'(x)w(x) + b'(-x)w(-x) = 0.$$

Similarly, we can consider Klein-Gordon equation.

Theorem 2.5 *Let w be the unit step function. The Laplace transform of the solution $u(x,t)$ of Klein-Gordon equation*

$$u_{tt} - u_{xx} + u(x,t) = 0$$

can be represented by

$$U(x,s) = \, A(s)e^{\sqrt{s^2+1}x} + B(s)e^{-\sqrt{s^2+1}x} - \frac{s\alpha+\beta}{s^2+1}$$

for $u(x,0) = \alpha$ and $u_t(x,0) = \beta$.

Proof. Let us put $U(x,s) = \pounds[u(x,t)]$, $u(x,0) = \alpha$ and $u_t(x,0) = \beta$. Taking Laplace transform with respect to t, we have

$$s^2U - s\alpha - \beta - \frac{\partial^2 U}{\partial x^2} + U = 0.$$

Collecting the U-terms, we have

$$\frac{\partial^2 U}{\partial x^2} - (s^2+1)U = -s\alpha - \beta.$$

Since this equation may be regraded as ODE for $U(x,s)$ considered as a function of x, a general solution is

$$\begin{aligned} U(x,s) &= A(s)e^{\sqrt{s^2+1}x} + B(s)e^{-\sqrt{s^2+1}x} \\ &+\frac{(s\alpha+\beta)}{2\sqrt{s^2+1}}e^{\sqrt{s^2+1}x}\int e^{-\sqrt{s^2+1}x}\,dx \\ &-\frac{(s\alpha+\beta)}{2\sqrt{s^2+1}}e^{-\sqrt{s^2+1}x}\int e^{\sqrt{s^2+1}x}\,dx \\ &= A(s)e^{\sqrt{s^2+1}x} + B(s)e^{-\sqrt{s^2+1}x} - \frac{s\alpha+\beta}{s^2+1} \end{aligned}$$

for the Wronskian $W = 2\sqrt{s^2+1}$.

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