Asymmetric Ascoli-type Theorems and Filter Exhaustiveness

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Abstract

We prove an Ascoli-type theorem, giving a necessary and sufficient condition for forward compactness of sets of functions, defined and with values in asymmetric metric spaces. Furthermore, we pose some open problems.

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1 Introduction

In the literature there have been many recent extensions of the classical Ascoli theorems (see [17, 19]) in the metric space context along several directions.
For example, in [16], some Ascoli-type theorems are proved, in connection with various kinds of convergence and exhaustiveness of function nets. In [1] and [4] these convergences, together with the concept of exhaustiveness, are considered in the context of filter/ideal convergence, and in this setting some Ascoli-type theorems in the metric space context are extended. In [15] some Ascoli-type theorem is proved, when the involved distance function is not required to be necessarily symmetric. Asymmetric distance has different applications in several branches of Mathematics (see for instance [15] and the bibliography therein), and is connected also with the study of various semi-continuity properties and related topics (see also, for example, [2]). Filter convergence and filter exhaustiveness have many developments in the very recent literature, for instance in limit and decomposition theorems for measures (see also [5, 6, 7, 8, 9, 10, 11, 13, 14]). A comprehensive survey about these topics can be found in [12]. In this paper we prove an Ascoli-type theorem for the asymmetric case in the metric space setting, extending earlier results proved in [1, 15, 16], in which equicontinuity is replaced by filter exhaustiveness. This tool allows us to give a necessary and sufficient condition for relative (forward) compactness of suitable function sets. Finally, we pose some open problems.

2 Preliminaries

We begin with recalling some basic notions on ideals and filters of \( \mathbb{N} \).

Let \( \mathcal{P}(\mathbb{N}) \) be the class of all subsets of \( \mathbb{N} \). A family \( \mathcal{I} \subset \mathcal{P}(\mathbb{N}) \) is called an ideal of \( \mathbb{N} \) iff \( A \cup B \in \mathcal{I} \) whenever \( A, B \in \mathcal{I} \) and for each \( A \in \mathcal{I} \) and \( B \subset A \) we get \( B \in \mathcal{I} \). A class of sets \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \) is a filter of \( \mathbb{N} \) iff \( A \cap B \in \mathcal{F} \) for all \( A, B \in \mathcal{F} \) and for every \( A \in \mathcal{F} \) and \( B \supset A \) we have \( B \in \mathcal{F} \).

An ideal \( \mathcal{I} \) (resp. a filter \( \mathcal{F} \)) of \( \mathbb{N} \) is said to be non-trivial iff \( \mathcal{I} \neq \emptyset \) and \( \mathbb{N} \not\in \mathcal{I} \) (resp. \( \mathcal{F} \neq \emptyset \) and \( \emptyset \not\in \mathcal{F} \)). A non-trivial ideal \( \mathcal{I} \) of \( \mathbb{N} \) is said to be admissible iff it contains all the single sets.

Given an ideal \( \mathcal{I} \) of \( \mathbb{N} \), we call dual filter of \( \mathcal{I} \) the family \( \mathcal{F} = \{ \mathbb{N} \setminus I : I \in \mathcal{I} \} \). In this case we say that \( \mathcal{I} \) is the dual ideal of \( \mathcal{F} \) and we get \( \mathcal{I} = \{ \mathbb{N} \setminus F : F \in \mathcal{F} \} \). A non-trivial filter \( \mathcal{F} \) of \( \mathbb{N} \) is free iff its dual ideal is admissible.

A filter \( \mathcal{F} \) of \( \mathbb{N} \) is called a \( P \)-filter iff for every sequence \( (A_n)_n \) in \( \mathcal{F} \) there exists another sequence \( (B_n)_n \) in \( \mathcal{F} \), such that the symmetric difference \( A_n \triangle B_n \) is finite for all \( n \in \mathbb{N} \) and \( \bigcap_{n=1}^{\infty} B_n \in \mathcal{F} \).

The filter \( \mathcal{F}_{\text{cofin}} \) is the filter of all subsets of \( \mathbb{N} \) whose complement is finite, and its dual ideal \( \mathcal{I}_{\text{fin}} \) is the family of all finite subsets of \( \mathbb{N} \). The filter \( \mathcal{F}_{\text{st}} \) is the filter of all subsets of \( \mathbb{N} \) having asymptotic density one, while its dual ideal \( \mathcal{I}_{\text{st}} \) is the family of all subsets of \( \mathbb{N} \) with asymptotic density zero. Note that
both $F_{\text{cofin}}$ and $F_{st}$ are $P$-filters (see also [12]).

We now recall the main notions and properties about convergence, closure, compactness and exhaustiveness in the filter setting and in the asymmetric case (see also [4, 15]).

An asymmetric metric space $X = (X, d)$ is any nonempty set endowed with an asymmetric metric or asymmetric distance $d : X \times X \to \mathbb{R}$, satisfying the following properties:

- $d(x, y) \geq 0$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- for each $x, y, z \in X$ we get $d(x, z) \leq d(x, y) + d(y, z)$.

Given an element $x_0 \in X$ and a positive real number $r$, we call forward open ball (resp. backward open ball) of center $x_0$ and radius $r$ the set $B^+(x_0, r) := \{y \in X : d(x_0, y) < r\}$ (resp. $B^-(x_0, r) := \{y \in X : d(y, x_0) < r\}$).

Let $F$ be a free filter of $\mathbb{N}$ and choose $\pi \in X$. A sequence $(s_n)_n$ in $X$ is said to be $F$-forward bounded (resp. $F$-backward bounded) iff there is $r > 0$ such that $\{n \in \mathbb{N} : d(\pi, s_n) \leq r\} \in F$ (resp. $\{n \in \mathbb{N} : d(s_n, \pi) \leq r\} \in F$). We say that $(s_n)_n$ is forward bounded (resp. backward bounded) iff it is $F_{\text{cofin}}$-forward bounded (resp. $F_{\text{cofin}}$-backward bounded).

Let $x \in X$. A sequence $x_n$, $n \in \mathbb{N}$, in $X$ is $F$-forward convergent (resp. $F$-backward convergent) to $x \in X$ iff

$$\{n \in \mathbb{N} : d(x, x_n) \leq \varepsilon\} \in F \quad (\text{resp.} \quad \{n \in \mathbb{N} : d(x_n, x) \leq \varepsilon\} \in F)$$

for every $\varepsilon > 0$. We say that $(x_n)_n$ forward (resp. backward) converges to $x$ iff $F_{\text{cofin}}$-forward (resp. $F_{\text{cofin}}$-backward) converges to $x$.

Let $(X, d_X)$ and $(Y, d_Y)$ be two asymmetric metric spaces, and choose $\overline{y} \in Y$. We say that a sequence $f_n : X \to Y$, $n \in \mathbb{N}$, is $F$-pointwise forward (resp. backward) bounded, iff for every $x \in X$ there are a set $F_x \in F$ and a positive real number $k_x$ with

$$f_n(x) \in B^+(\overline{y}, k_x) \quad (\text{resp.} \quad f_n(x) \in B^-(\overline{y}, k_x)) \quad \text{for each} \quad n \in F_x. \quad (1)$$

Throughout this paper, the concept of (filter forward/backward) compactness is always intended as sequential compactness (for various definitions of compactness and related comparisons, see also [4, 15, 18, 20]).

A subset $S \subset X$ is $F$-forward compact (resp. $F$-backward compact) iff every sequence in $S$ has a subsequence, $F$-forward (resp. $F$-backward) convergent to an element of $S$. We say that $S \subset X$ is forward compact (resp. backward compact) iff it is $F_{\text{cofin}}$-forward compact (resp. $F_{\text{cofin}}$-backward compact).

Given $S \subset X$, the $F$-forward (resp. $F$-backward) closure of $S$ in the set of all elements $x \in X$, such that there is a sequence $(s_n)_n$ in $S$, $F$-forward (resp. $F$-backward) convergent to $x$. 


Remark 2.1. Observe that, by proceeding analogously as in [3, Proposition 2.4], it is possible to prove that every $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) convergent sequence in an asymmetric metric space $(X, d)$ has a forward (resp. backward) convergent subsequence. From this it follows that, for every free filter $\mathcal{F}$ of $\mathbb{N}$, $\mathcal{F}$-forward ($\mathcal{F}$-backward) compactness and closure are equivalent to forward (backward) compactness and closure respectively (see also [4]).

Let $(X, d_X)$ and $(Y, d_Y)$ be two asymmetric metric spaces, and $x \in X$. A sequence $f_n : X \to Y$, $n \in \mathbb{N}$, is said to be $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) exhaustive at $x$ iff for every $\varepsilon > 0$ there exist $\delta > 0$ and a set $F \in \mathcal{F}$ (depending on $\varepsilon$ and $x$) with $d_Y(f_n(x), f_n(y)) < \varepsilon$ (resp. $d_Y(f_n(y), f_n(x)) < \varepsilon$) whenever $n \in F$ and $y \in X$ with $d_X(x, y) < \delta$. We say that $(f_n)_n$ is $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) exhaustive on $X$ iff it is $\mathcal{F}$-forward (resp. $\mathcal{F}$-backward) exhaustive at every $x \in X$.

We say that a set $C \subset Y^X$ is forward (resp. backward) equicontinuous on $X$ iff for every $\varepsilon > 0$ and $x \in X$ there is $\delta > 0$ (depending on $\varepsilon$ and $x$) such that for every $f \in C$ and $y \in X$ with $d_X(x, y) < \delta$ we get $d_Y(f(x), f(y)) < \varepsilon$ (resp. $d_Y(f(y), f(x)) < \varepsilon$). Observe that equicontinuity is in general strictly stronger than filter exhaustiveness (see also [4, Remark 3.9]).

Let $C(X, Y)$ denote the sets of all functions $f \in Y^X$ with the property that for every $\varepsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that $f(y) \in B^+(f(x), \varepsilon)$ whenever $y \in B^+(x, \delta)$.

The uniform metric $\overline{p}$ on $Y^X$ is defined by

$$\overline{p}(f, h) = \sup \{d_Y(f(x), h(x)) : x \in X\},$$

where $\overline{d_Y}(a, b) = \min\{d_Y(a, b), 1\}$, $a, b \in Y$. We say that a subset $C \subset Y^X$ is forward (resp. backward) totally bounded iff, for every $\varepsilon > 0$, $C$ is contained in the union of a finite number of forward (resp. backward) open balls of radius $\varepsilon$ in the metric $\overline{p}$.

3 The main results

We begin with recalling the following

Proposition 3.1. (see [15, Proposition 5.5]) Let $(X, d_X)$ and $(Y, d_Y)$ be two asymmetric metric and forward compact spaces, and let $C \subset C(X, Y)$ be forward equicontinuous. Then $C$ is forward totally bounded with respect to $\overline{p}$.

Proposition 3.2. (see [15, Proposition 5.11]) Let $(X, d_X)$ and $(Y, d_Y)$ be two asymmetric metric spaces such that $Y$ is forward compact and forward convergence implies backward convergence in $Y$. Then $C(X, Y)$ is complete with respect to $\overline{p}$. 
We now prove the next result.

**Proposition 3.3.** Let \((X, d_X)\) and \((Y, d_Y)\) be asymmetric metric spaces such that \(X\) is forward compact, \(\mathcal{F}\) be a free filter of \(\mathbb{N}\), \(\mathcal{G}\) be as in (1), and assume that forward closed and forward bounded sets in \(Y\) are forward compact. Let \(f_n : X \to Y, n \in \mathbb{N}\), be a function sequence, \(\mathcal{F}\)-forward exhaustive and \(\mathcal{F}\)-pointwise forward bounded. Then there exist \(F \in \mathcal{F}\) and a forward compact set \(Z \subset Y\) with \(f_n(X) \subset Z\) for every \(n \in F\).

**Proof:** Choose arbitrarily \(a \in X\). By \(\mathcal{F}\)-forward exhaustiveness of \((f_n)\), in correspondence with \(a\) and \(\varepsilon = 1\) there exist \(\delta_a > 0\) and \(F_a \in \mathcal{F}\) with \(d_Y(f_n(a), f_n(x)) < 1\) whenever \(x \in B^+(a, \delta_a)\) and \(n \in F_a\). Since \(X\) is forward compact, there is a finite cover \(\mathcal{V}\) of \(X\), \(\mathcal{V} = \{B^+(a_j, \delta_{a_j}) : j = 1, 2, \ldots, q\}\) (see also [18], [20, Theorem III]). Since \((f_n)\) is \(\mathcal{F}\)-pointwise forward bounded, there are \(F' \in \mathcal{F}\) and \(k' > 0\) such that \(f_n(a_j) \subset B^+(\mathcal{G}, k')\) for every \(n \in F'\) and \(j \in [1, q]\). Let \(F = F' \cap \left(\bigcap_{j=1}^{q} F_{a_j}\right)\), and choose \(n \in F\) and \(x \in X\). Note that \(F \in \mathcal{F}\). There is \(j \in [1, q]\) with \(x \in B^+(a_j, \delta_{a_j})\), and hence we get

\[
d_Y(\mathcal{G}, f_n(x)) \leq d_Y(\mathcal{G}, f_n(a_j)) + d_Y(f_n(a_j), f_n(x)) < k' + 1.
\]

Therefore, \(f_n(X) \subset B^+(\mathcal{G}, k' + 1)\) for each \(n \in F\). If \(Z\) is the forward closure of \(B^+(\mathcal{G}, k' + 1)\) with respect to \(d_Y\), then \(Z\) is forward compact, since by hypothesis forward closed and forward bounded sets in \(Y\) are forward compact. \(\square\)

We now turn to our main theorem, which extends [1, Theorem 3.7], [15, Theorem 5.12] and [16, Theorems 3.2.19, 3.2.20] to the setting of filter exhaustiveness and asymmetric distance, and gives a necessary and sufficient condition for (forward) relative compactness of function sets.

**Theorem 3.4.** Let \((X, d_X)\) and \((Y, d_Y)\) be asymmetric metric spaces, such that \(X\) is forward compact, \(\mathcal{G}\) be as in (1), and \(\mathcal{F}\) be a \(\mathcal{P}\)-filter of \(\mathbb{N}\). Assume that every forward closed and \(\mathcal{F}\)-forward bounded subset of \(Y\) is \(\mathcal{F}\)-forward compact, and forward convergence implies backward convergence in \(Y\). Let \(\overline{\mathcal{P}}\) be as in (2), and \(\mathcal{C} \subset C(X, Y)\) be such that

1. every sequence \((f_n)\) in \(\mathcal{C}\) has a subsequence \((f_{n_r})_r\), \(\mathcal{F}\)-pointwise forward bounded in \(Y\).

Suppose moreover that

2. every sequence in \(\mathcal{C}\), pointwise \(\mathcal{F}\)-forward convergent in \(Y^X\), has a \(\mathcal{F}\)-forward exhaustive subsequence.
Then the set $\overline{C}$, that is the forward closure of $C$ with respect to $\overline{p}$, is forward compact.

Conversely, if $\overline{C}$ is forward compact with respect to $\overline{p}$, then 3.4.1) and 3.4.2) hold.

**Proof:** For each $x \in X$, set $C_x := \{f(x) : f \in C\}$, and let $\overline{C}_x$ be the forward closure of $C_x$ in $Y$. We claim that $\overline{C}_x$ is forward compact in $Y$. Indeed, choose $y \in \overline{C}_x$. There is a sequence $(y_n)_n$ in $C_x$, forward (and also backward) convergent to $y$ with respect to $d_Y$. So, in correspondence with $\varepsilon = 1$ there is a natural number $n_0$ with $d_Y(y, y_n) \leq 1$ whenever $n \geq n_0$. Moreover there is a sequence $(f_n)_n$ in $C$ such that $f_n(x) = y_n$ for each $n \in \mathbb{N}$. By 3.4.1), there exist a subsequence $(f_{n_k})_k$ of $(f_n)_n$, a positive real number $k_x$ and a set $F_x \in \mathcal{F}$ with $y_{n_k} \in B^+(\overline{y}, k_x)$ whenever $r \in F_x$. Thus there is a positive integer $r_0$ with

$$d_Y(\overline{y}, y) \leq d_Y(\overline{y}, y_{n_{r_0}}) + d_Y(y_{n_{r_0}}, y) \leq k_x + 1,$$

getting forward boundedness of $\overline{C}_x$ and hence also $\mathcal{F}$-forward compactness and forward compactness of $\overline{C}_x$, thanks to the hypotheses and Remark 2.1.

Now, since $\overline{C}_x$ is forward compact in $Y$ and forward convergence implies backward convergence in $Y$, then, by the Tychonoff theorem, the set $\Pi_{x \in X} \overline{C}_x$ is compact in $Y^X$ with respect to the pointwise convergence. Since $C \subseteq \Pi_{x \in X} \overline{C}_x$, then we get that every sequence $(f_n)_n$ in $C$ has a subsequence $(f_{n_k})_k$, pointwise convergent to a suitable function $h \in Y^X$ and hence pointwise bounded too.

Pick arbitrarily any sequence $(f_n)_n$ in $C$. We will prove that $(f_n)_n$ has a subsequence, forward convergent with respect to $\overline{p}$. Let $(f_{n_k})_k$ be as above. From 3.4.2) and the above argument it follows that $(f_{n_k})_k$ admits a pointwise bounded and $\mathcal{F}$-forward exhaustive subsequence, say $(g_n)_n$. By Proposition 3.3 there exist a compact subset $Z \subseteq Y$ and a set $F_0 \subseteq \mathcal{F}$, with $\{g_n : n \in F_0\} \subseteq C(X, Z)$. Note that even the sequence $(g_n)_{n \in F_0}$ is pointwise convergent to $h$, and so by 3.4.2) it has an $\mathcal{F}$-forward exhaustive subsequence, say $(h_n)_n$. Now observe that, arguing analogously as in [15, Lemma 5.2], it is possible to show that $\mathcal{F}$-forward exhaustiveness implies $\mathcal{F}$-backward exhaustiveness, because $Z$ is forward compact, forward convergence implies backward convergence in $Z$ with respect to $d_Y$, and the forward and backward topologies on $Z$ are equivalent. By virtue of $\mathcal{F}$-forward exhaustiveness of the sequence $(h_n)_n$ and since $\mathcal{F}$ is a $P$-filter, arguing analogously as in [1, Lemma 3.6] it is possible to find a set $F^* \subseteq \mathcal{F}$, such that for every $\varepsilon > 0$ there are $\delta > 0$ and $n_0 \in F^*$ such that $d_Y(h_n(x), h_n(z)) < \varepsilon$ for any $x, z \in X$ with $d_X(x, z) < \delta$ and $n \in F^*, n \geq n_0$. From this, since $h_n \in C(X, Z)$ for every $n \in \mathbb{N}$, we get that the sequence $(h_n)_{n \in F^*}$ is forward equicontinuous on $X$. By Proposition 3.1 it follows that the set $\{h_n : n \in F^*\}$ is forward totally bounded with respect to $\overline{p}$. By Proposition 3.2, since $Z$ is forward compact, we get that $C(X, Z)$ is complete with respect to $\overline{p}$. If $Z$ denotes the closure of $\{h_n : n \in F^*\}$ in
\( C(X,Z) \) with respect to \( \bar{\rho} \), then we get that \( Z \) is complete and forward totally bounded. Thus, arguing analogously as in [15, Theorem 4.8], it follows that \( Z \) is forward compact. Thus the sequence \( (h_n)_{n \in F^*} \), and so a fortiori the sequence \( (f_n)_n \), has a subsequence, forward convergent with respect to \( \bar{\rho} \), getting forward compactness of the set \( \bar{C} \).

We now turn to the last part. Let \( (f_n)_n \) be a sequence of functions in \( C \). Since \( \bar{C} \) is forward compact, there is a subsequence \( (f_{n_k})_k \) of \( (f_n)_n \), convergent to a function \( f_0 \in Y^X \) with respect to the metric \( \bar{\rho} \) in (2), and hence also pointwise convergent with respect to \( d_Y \), getting forward compactness of \( \bar{C}_x \) in \( Y \) for every \( x \in X \). From this it follows that \( (f_{n_k}(x))_k \) is forward totally bounded for each \( x \in X \) (see also [15, Proposition 4.8]), and hence \( F \)-pointwise forward bounded too, getting 3.4.1). Moreover, since \( \bar{C} \) is forward compact, \( \bar{C} \cup \{ f_0 \} \) is forward totally bounded, and in particular for every \( \varepsilon > 0 \) there are \( h_1, \ldots, h_q \in \bar{C} \) with the property that for every \( n \geq 0 \) there is \( i \in [1,q] \) with \( \bar{\rho}(h_i, f_n) < \varepsilon/2 \), and hence \( d_Y(h_i(x), f_n(x)) < \varepsilon/2 \) for each \( x \in X \). Let now \( i \in [1,q] \). Since \( h_i \in C(X,Y) \) and \( X \) is forward compact, then \( h_i(X) \) is forward compact in \( Y \), and hence forward totally bounded too. Therefore for every \( \varepsilon > 0 \) there are \( y_{i,1}, y_{i,2}, \ldots, y_{i,p(i)} \in Y \) such that for every \( x \in X \) there is \( j \in [1,p(i)] \) with \( d_Y(y_{i,j}, h_i(x)) < \varepsilon/2 \). Let now

\[
A = \{(i,j) : i \in [1,q], j \in [1,p(i)]\}. \tag{3}
\]

Note that \( A \) is a finite subset of \( \mathbb{N}^2 \). For each \( \beta \in A \), let \( y_\beta = y_{i,j} \), where \( i, j \) are as in (3). Thus for every \( n \geq 0 \) and \( x \in X \) there is \( \beta \in A \) with

\[
d_Y(y_\beta, f_n(x)) \leq d_Y(y_\beta, h_i(x)) + d_Y(h_i(x), f_n(x)) < \varepsilon,
\]

getting forward total boundedness of the set \( E = \{ f_n(x) : x \in X, n \geq 0 \} \).

So, \( E \) is forward bounded. Arguing analogously as in Proposition 3.3, it is possible to see that the forward closure \( Z \) of \( E \) with respect to \( d_Y \) is forward bounded, and hence also forward compact, by hypothesis. By [15, Lemma 4.2], the forward and backward convergence on \( Z \) coincide. By [15, Lemma 5.8], \( f \in C(X,Z) \). Furthermore note that, by [15, Lemma 5.2], in the space \( Z^X \) with the metric \( \bar{\rho} \), forward and backward convergence coincide.

We now prove forward equicontinuity of the sequence \( (f_{n_k})_k \) on \( X \), which will imply its \( F \)-forward exhaustiveness with respect to any free filter \( F \) of \( \mathbb{N} \), getting in particular 3.4.2). Choose arbitrarily \( x \in X \). Since \( f_0 \in C(X,Z) \), in correspondence with \( \varepsilon > 0 \) and \( x \in X \) there is \( \delta > 0 \) with \( d_Y(f_0(x), f_0(z)) < \varepsilon/3 \) whenever \( z \in B^+(x, \delta) \). By convergence of \( (f_{n_k})_k \) to \( f_0 \) on \( X \) with respect to \( \bar{\rho} \), there is \( \overline{k} \in \mathbb{N} \) with

\[
d_Y(f_0(z), f_{n_k}(z)) < \varepsilon/3 \quad \text{and} \quad d_Y(f_{n_k}(x), f_0(x)) < \varepsilon/3 \quad \text{for each} \quad k \geq \overline{k}. \tag{4}
\]

From (4) we get

\[
d_Y(f_{n_k}(x), f_{n_k}(z)) \leq d_Y(f_{n_k}(x), f_0(x)) + d_Y(f_0(x), f_0(z)) + d_Y(f_0(z), f_{n_k}(x)) < \varepsilon \tag{5}
\]
for every $k \geq \bar{k}$ and $z \in B^+(x,\delta)$. Forward equicontinuity of $(f_{nk})_k$ follows from (5), since the $f_{nk}$’s belong to $C(X,Z)$. This ends the proof. □

**Remarks 3.5.** (a) Note that, in general, condition 3.4.2) is strictly weaker than forward/backward equicontinuity (see also [16, Remark 3.2.17]), and that a condition similar to 3.4.2) was given in [16, Theorems 3.2.19, 3.2.20].

(b) Observe that in the asymmetric case, under the hypotheses of Theorem 3.4, in general forward compactness of the set $\mathcal{C}$ does not imply forward equicontinuity of $\mathcal{C}$ (see also [15, Example 3.5]). Moreover, in general the condition that forward convergence implies backward convergence cannot be dropped (see also [15, Example 5.13]).

**Open problems:** (a) Prove some Ascoli-type theorems for functions, defined and/or taking values in spaces endowed with other structures.

(b) Investigate some other kinds of (filter) exhaustiveness in abstract contexts.

**References**


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