On $\lambda$-Bernoulli Polynomials of the Second Kind

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Abstract

In this paper, we study the $\lambda$-analogues of Bernoulli polynomials of the second kind, and we derive some new identities related to those polynomials.

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1. Introduction

For \( r \in \mathbb{N} \), the Bernoulli polynomials of order \( r \) are defined by the generating function to be
\[
\left( \frac{t}{e^t-1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \text{ (see [1-12]).} \tag{1.1}
\]

L. Carlitz have introduced the degenerate Bernoulli polynomials of order \( r \) as follows:
\[
\left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \right)^r (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(r)}(x|\lambda) \frac{t^n}{n!}, \text{ (see [2]).} \tag{1.2}
\]

When \( x = 0 \), \( \beta_n^{(r)}(\lambda) = \beta_n^{(r)}(0|\lambda) \) are called the degenerate Bernoulli numbers. The Bernoulli polynomials of the second kind are defined by the generating function to be
\[
\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \text{ (see [10]).} \tag{1.3}
\]

When \( x = 0 \), \( b_n = b_n(0) \) are called the \( n \)-th Bernoulli numbers of the second kind.

It is well known that
\[
\left( \frac{t}{\log(1+t)} \right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x) \frac{t^n}{n!}, \tag{1.4}
\]

From (1.3) and (1.4), we have \( b_n(x) = B_n^{(n)}(x), \ (n \geq 0) \).

Recently, Korobov introduced the special polynomials which are given by the generating function to be
\[
\frac{\lambda t}{(1+t)^{\frac{1}{\lambda}}-1} (1+t)^x = \sum_{n=0}^{\infty} K_n(x|\lambda) \frac{t^n}{n!}, \text{ (see [9]).} \tag{1.5}
\]

Note that \( \lim_{n \to 0} K_n(x|\lambda) = b_n(x), \ (n \geq 0) \). When \( x = 0 \), \( K_n(\lambda) = K_n(0|\lambda) \) are called Korobov numbers.

In this paper, we consider the \( \lambda \)-analogue of Bernoulli polynomials of the second kind and give some new identities of those polynomials.

2. \( \lambda \)-analogues of Bernoulli polynomials of the second kind

For \( \lambda \in [0,1] \), we consider the \( \lambda \)-analogue of Bernoulli polynomials of the second kind which are given by definite integral as follows:
\[
\int_0^1 (1+t)^{\lambda y+x} dy = \frac{1}{\log(1+t)} \left( \frac{(1+t)^{\lambda}-1}{\lambda} \right) (1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{2.1}
\]
Note that \( b_{n,1}(x) = b_n(x) \) and \( \lim_{\lambda \to 0} b_{n,\lambda}(x) = (x)_n, \) \((n \geq 0)\), where \( (x)_n = \sum_{l=0}^{n} S_1(n,l) x^l \) with \( S_1(n,l) \) Stirling number. When \( x = 0 \), \( b_{n,\lambda} = b_{n,\lambda}(0) \) are called \( \lambda \)-Bernoulli numbers of the second kind.

From (2.1), we note
\[
\int_0^1 (\lambda y + x)_n dy = b_{n,\lambda}(x), \quad (n \geq 0),
\]
where \((x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l \). It is easy to show that
\[
\int_0^1 (1+t)^{x+y} dy = \frac{t}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}.
\]
Thus, by (2.3), we get
\[
\int_0^1 (x+y)_n dy = b_n(x), \quad (n \geq 0).
\]
We observe that
\[
\int_0^1 (\lambda y + x)_n dy = \lambda^n \int_0^1 \left( y + \frac{x}{\lambda} \right)_n dy = \lambda^n b_n \left( \frac{x}{\lambda} \right).
\]
Therefore, by (2.2) and (2.5), we get
\[
b_{n,\lambda}(x) = \lambda^n b_n \left( \frac{x}{\lambda} \right), \quad (n \geq 0).
\]
From (2.2), we have
\[
\int_0^1 (1+t)^{\lambda y + x} dy = \frac{t}{\log(1+t)} \left( \frac{(1+t)^\lambda - 1}{\lambda t} \right) (1+t)^x.
\]
It is not difficult to show that
\[
\frac{1}{\lambda t} \left( (1+t)^\lambda - 1 \right) = \frac{1}{\lambda^2} \sum_{n=1}^{\infty} \left( \frac{\lambda}{n} \right) t^n = \sum_{n=0}^{\infty} \frac{1}{\lambda} \left( \frac{\lambda}{n+1} \right) t^n.
\]
By (2.7) and (2.8), we get
\[
\frac{t}{\log(1+t)} \left( \frac{(1+t)^\lambda - 1}{\lambda t} \right) (1+t)^x
= \left( \sum_{l=0}^{\infty} b_l(x) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \frac{1}{\lambda} \left( \frac{\lambda}{m+1} \right) t^m \right)
= \left( \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} b_l(x) \left( \frac{\lambda}{n-l+1} \right) \frac{1}{\lambda} \frac{n!(n-l)!}{l!(n-l)!} \right) t^n \right) \frac{1}{n!}
= \left( \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{n!}{l!} b_l(x) \left( \frac{\lambda}{n-l+1} \right) \frac{1}{\lambda} \right) t^n \right) \frac{1}{n!}.
\]
Therefore, by (2.1) and (2.9), we obtain the following theorem.
Theorem 2.1. For $n \geq 0$, we have
\[ b_{n,\lambda}(x) = \sum_{l=0}^{\infty} \left( \begin{array}{c} n \\ l \end{array} \right) (n-l)! b_l(x) \frac{\lambda}{n-l+1}. \]

In particular,
\[ b_{n,\lambda}(x) = \lambda^n b_n \left( \frac{x}{\lambda} \right). \]

Corollary 2.2. For $n \geq 0$, we have
\[ \sum_{l=0}^{n} \frac{S_1(n,l)}{l+1} ((x+\lambda)^{l+1} - x^{l+1}) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (n-l)! b_l(x) \left( \begin{array}{c} \lambda \\ n-l+1 \end{array} \right). \]

From (1.3) and (2.7), we have
\[ \sum_{n=0}^{\infty} \frac{b_n(x)^t}{n!} = \frac{t}{\log(1+t)} (1+t)^x = \frac{\lambda t}{(1+t)^\lambda - 1} \int_{0}^{1} (1+t)^{\lambda y+x} dy \]
\[ = \left( \sum_{l=0}^{\infty} K_l(\lambda) t^l \right) \left( \sum_{m=0}^{\infty} \int_{0}^{1} (\lambda y + x)_m dy \frac{t^m}{m!} \right) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) K_m(\lambda) b_{n-m,\lambda}(x) \right) \frac{t^n}{n!}. \]

Therefore, by (2.10), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have
\[ b_n(x) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) K_m(\lambda) b_{n-m,\lambda}(x). \]

By replacing $t$ by $e^t - 1$ in (2.1), we get
\[ \frac{1}{\lambda t} (e^{\lambda t} - 1) e^{xt} = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{1}{n!} (e^t - 1)^n \]
\[ = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \]
\[ = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} b_{n,\lambda}(x) S_2(m,n) \right) \frac{t^m}{m!}. \]

We observe that
\[ \frac{1}{\lambda t} (e^{\lambda t} - 1) e^{xt} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} x^m \frac{\lambda^{m-n}}{m-n+1} \left( \begin{array}{c} m \\ l \end{array} \right) \right) \frac{t^m}{m!}. \]

Therefore, by (2.11) and (2.12), we obtain the following theorem.
Theorem 2.4. For \( m \geq 0 \), we have
\[
\sum_{n=0}^{m} \frac{\lambda^{m-n}}{m-n+1} \binom{m}{l} x^n = \sum_{n=0}^{m} b_{n,\lambda}(x) S_2(m, n).
\]

Let us consider the multivariate integral on \([0, 1]\) given by
\[
\int_0^1 \cdots \int_0^1 (1 + t)^{\lambda(x_1 + \cdots + x_r) + x} \, dx_1 \cdots dx_r.
\] (2.13)

From (2.1), we note that
\[
\int_0^1 \cdots \int_0^1 (1 + t)^{\lambda(x_1 + \cdots + x_r) + x} \, dx_1 \cdots dx_r
= \left( \frac{1}{\log(1 + t)} \frac{(1 + t)^{\lambda} - 1}{\lambda} \right)^r (1 + t)^x.
\] (2.14)

Now, we define \( \lambda \)-Bernoulli polynomials of the second kind with order \( r \) as follows:
\[
\left( \frac{1}{\log(1 + t)} \frac{(1 + t)^{\lambda} - 1}{\lambda} \right)^r (1 + t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}.
\] (2.15)

Now, we observe that
\[
\left( \frac{1}{\log(1 + t)} \frac{(1 + t)^{\lambda} - 1}{\lambda} \right)^r (1 + t)^x
= \left( \frac{t}{\log(1 + t)} \right)^r \left( \frac{(1 + t)^{\lambda} - 1}{\lambda t} \right)^r (1 + t)^x
= \left( \sum_{l=0}^{\infty} B_l^{(l-r+1)}(x+1) \frac{t^l}{l!} \right)
\times \left( \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_r=m} \frac{\lambda}{(m-l_1+1) \cdots (m-l_r+1)} \binom{m}{l_1} \cdots \binom{m}{l_r} \frac{t^m}{m!} \right)
= \sum_{n=0}^{\infty} \left( \sum_{\lambda,m=0}^{\infty} \binom{n}{m} \sum_{l_1+\cdots+l_r=m} \frac{\lambda^{r-m(l_1+\cdots+l_r)}}{(m-l_1+1) \cdots (m-l_r+1)} \binom{m}{l_1} \cdots \binom{m}{l_r} B_{n-m-r+1}^{(n-m-r+1)}(x+1) \right) \frac{t^m}{m!}.
\] (2.16)

Therefore, by (2.15) and (2.16), we obtain the following theorem.

Theorem 2.5. For \( n \geq 0 \), we have
\[
b_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n} \binom{n}{m} \sum_{l_1+\cdots+l_r=m} \frac{\lambda^{r-m(l_1+\cdots+l_r)}}{(m-l_1+1) \cdots (m-l_r+1)} \binom{m}{l_1} \cdots \binom{m}{l_r} B_{n-m-r+1}^{(n-m-r+1)}(x+1).
\]
By replacing $t$ by $e^t - 1$ in (2.15), we get

$$
\left( \frac{1}{\lambda t} (e^{\lambda t} - 1) \right)^r e^{xt} = \sum_{m=0}^{\infty} b_{m, \lambda}(x) \frac{1}{m!} (e^t - 1)^m
$$

$$
= \sum_{m=0}^{\infty} b_{m, \lambda}^{(r)}(x) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^m}{m!}
$$

$$
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} b_{m, \lambda}^{(r)}(x) S_2(n, m) \right) \frac{t^m}{m!}.
$$

(2.17)

Note that

$$
\left( \frac{1}{\lambda t} (e^{\lambda t} - 1) \right)^r e^{xt} = \left( \frac{1}{\lambda t} \right)^r r! \left( \sum_{l=r}^{\infty} S_2(l, r) \frac{\lambda^l t^l}{l!} \right) e^{xt}
$$

$$
= \left( \sum_{l=0}^{\infty} S_2(l + r, r) \frac{r!}{(l + r)!} \frac{\lambda^l t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \right)
$$

$$
= \left( \sum_{l=0}^{\infty} S_2(l + r, r) \frac{1}{(l + r)!} \frac{\lambda^l t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \right)
$$

$$
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_2(l + r, r) \frac{1}{(l + r)!} \frac{\lambda^l x^{n-l}}{l!} \right) \frac{t^n}{n!}.
$$

(2.18)

Therefore, by (2.17) and (2.18), we obtain the following theorem.

**Theorem 2.6.** For $n \geq 0$, we have

$$
\sum_{m=0}^{n} b_{m, \lambda}^{(r)}(x) S_2(m, n) = \sum_{m=0}^{n} S_2(m + r, r) \left( \frac{n}{m+r} \right) \lambda^m x^{n-m}.
$$

From (2.14) and (2.15), we can derive the following equation:

$$
\int_0^1 \binom{\lambda y + x}{n} dy = b_{n, \lambda}(x), \quad (n \geq 0),
$$

and

$$
\int_0^1 \cdots \int_0^1 \binom{\lambda (x_1 + \cdots + x_r) + x}{n} dx_1 \cdots dx_r = b_{n, \lambda}^{(r)}(x).
$$

**REFERENCES**


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