Structure of $BF$-algebras

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Abstract

In this paper, we investigate some properties of $BF$-algebras for a homomorphism of $BF$-algebras.

Mathematics Subject Classification: 03G25

Keywords: $BF$-algebra, subalgebra, normal subalgebra.

1 Introduction

Y. B. Jun, E. H. Roh and H. S. Kim ([3]) introduced a $BH$-algebra. They defined the notion of an ideal and a boundedness in a $BH$-algebra and studied some properties of them. C. B. Kim and H. S. Kim ([1]) defined a $BG$-algebra and considered some related properties. A. Walendziak ([4]) introduced the notion of $BF/BF_1/BF_2$-algebras and investigated some properties of (normal) ideals and subalgebras in $BF/BF_1/BF_2$-algebras. And he studied the properties and characterizations of them.

In this paper, we investigate some properties of $BF$-algebras for a homomorphism of $BF$-algebras.

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2 Preliminaries.

We review some definitions and results discussed in [4].

By a $BF$-algebra we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B3) $0 * (x * y) = y * x$

for all $x, y \in X$.

A $BF$-algebra $(X, *, 0)$ is called a $BF_1$-algebra if it satisfies the following identity:

(BG) $x = (x * y) * (0 * y)$ for all $x, y \in X$.

A $BF$-algebra $(X, *, 0)$ is called a $BF_2$-algebra if it satisfies the following identity:

(BH) $x * y = y * x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity, we also call $X$ a $BF$-algebra. If we can define a binary operation “$\leq$” by $x \leq y$ if and only if $x * y = 0$. A non-empty subset $A$ of a $BF$-algebra $X$ is called a subalgebra of $X$ if $x * y \in A$ for any $x, y \in A$.

A non-empty subset $A$ of a $BF$-algebra $X$ is said to be normal (or a normal subalgebra) ([2]) of $X$ if $(x * a) * (y * b) \in A$ for any $x * y, a * b \in A$. Note that any normal subalgebra $A$ of a $BF$-algebra $X$ is a subalgebra of $X$, but the converse need not be true (see [2]). A mapping $f : X \to Y$ of $BF$-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Lemma 2.1. If $X$ is a $BF$-algebra, then

(i) $0 * (0 * x) = x$, for all $x \in X$.
(ii) $0 * x = 0 * y$ implied $x = y$ for any $x, y \in X$.
(iii) if $x * y = 0$, then $y * x = 0$ for any $x, y \in X$.

Lemma 2.2. Let $X$ be a $BF$-algebra and let $N$ be a subalgebra of $X$. If $x * y \in N$ for any $x, y \in N$, then $y * x \in N$.

Corollary 2.3. ([2]) Let $X$ be a $BF$-algebra and let $N$ be a normal subalgebra of $X$. If $x * y \in N$ for any $x, y \in N$, then $y * x \in N$.

A $BG$-algebra $(X; *, 0)$ ([1]) is an algebra of type (2,0) satisfying (B1), (B2) and (BG).

Theorem 2.4 Let $X$ be a $BF_1$-algebra. Then
(i) $X$ is a $BG$-algebra.

(ii) $x \ast y = 0$ implies $x = y$ for any $x, y \in X$.

(iii) The right cancellation law holds in $X$, i.e., if $x \ast y = z \ast y$, then $x = z$ for any $x, y, z \in X$.

(iv) The left cancellation law holds in $X$, i.e., if $y \ast x = y \ast z$, then $x = z$ for any $x, y, z \in X$.

3 Problems of a homomorphism

In what follows, let $X$ be a $BF$-algebra unless otherwise specified.

Let $(X; \ast, 0)$ be a $BF$-algebra and let $N$ be a normal subalgebra of a $BF$-algebra $X$. Define a relation “$\sim_N$” on $X$ by $x \sim_N y$ if and only if $x \ast y \in N$ for any $x, y \in X$. Then $\sim_N$ is a congruence relation on $X$ ([2]). Denote $X/N := \{[x]_N|x \in X\}$, where $[x]_N := \{y \in X|y \sim_N x\}$. Define $[x]_N \ast [y]_N := [x \ast y]_N$. Then “$\ast'$” is well-defined, since $\sim_N$ is a congruence relation.

**Theorem 3.1.** ([2]) Let $N$ be a normal subalgebra of a $BF$-algebra $X$. Then $(X/N; \ast', [0]_N)$ is a $BF$-algebra.

The $BF$-algebra $X/N$ discussed in Theorem 3.1 is called the quotient $BF$-algebra of $X$ by $N$.

**Theorem 3.2.** Let $X,Y$ be $BF$-algebras and let $Z$ be a $BF_2$-algebra. Let $h : X \to Y$ be an epimorphism and $g : X \to Z$ be a homomorphism. If $\text{Ker}(h) \subseteq \text{Ker}(g)$, then there exists a unique homomorphism $f : Y \to Z$ satisfying $f \circ h = g$.

**Proof.** For any $y \in Y$, there exists an $x \in X$ such that $y = h(x)$, since $h$ is onto. Given an element $x \in X$, we put $z := g(x)$. Define a mapping $f : Y \to Z$ by $f(y) = z$. Then $f$ is well-defined. In fact, if $y = h(x_1) = h(x_2), x_1, x_2 \in X$, then $0 = h(x_1) \ast h(x_2) = h(x_1 \ast x_2)$. Hence $x_1 \ast x_2 \in \text{Ker}(h)$. Since $\text{Ker}(h) \subseteq \text{Ker}(g)$, we have $0 = g(x_1 \ast x_2) = g(x_1) \ast g(x_2)$. By a similar way, $0 = g(x_2 \ast x_1) = g(x_2) \ast g(x_1)$. Since $Z$ is a $BF_2$-algebra, we have $g(x_1) = g(x_2)$. This means that $f$ is well-defined. Clearly, $g(x) = f(h(x))$ for any $x \in X$.

Let $y_1, y_2 \in Y$. Then there exist $x_1, x_2 \in X$ such that $h(x_1) = y_1, h(x_2) = h_2$, since $h$ is an epimorphism. Hence we have

$$f(y_1 \ast y_2) = f(h(x_1) \ast h(x_2)) = f(h(x_1 \ast x_2)) = g(x_1 \ast x_2) = g(x_1) \ast g(x_2).$$

Hence $f$ is a homomorphism. The uniqueness of $f$ follows from that $h$ is an epimorphism. 

\[\square\]
**Theorem 3.3.** Let $X, Y$ and $Z$ be $BF$-algebras, and let $g : X \to Z$ be a homomorphism and let $h : Y \to Z$ be a monomorphism with $\text{Im}(g) \subseteq \text{Im}(h)$. Then there exists a unique homomorphism $f : X \to Y$ satisfying $h \circ f = g$.

*Proof.* For each $x \in X$, $g(x) \in \text{Im}(g) \subseteq \text{Im}(h)$. Since $h$ is a monomorphism, there exists a unique $y \in Y$ such that $h(y) = g(x)$. Define a map $f : X \to Y$ by $f(x) = y$. Then $h \circ f = g$.

If $x_1, x_2 \in X$, then $g(x_1 * x_2) = h(f(x_1 * x_2))$. Since $g$ is a homomorphism, we have $g(x_1 * x_2) = g(x_1) * g(x_2) = h(f(x_1)) * h(f(x_2))$. Hence $h(f(x_1 * x_2)) = h(f(x_1) * f(x_2))$. Since $h$ is a monomorphism, we get $f(x_1 * x_2) = f(x_1) * f(x_2)$. Therefore $f$ is a homomorphism. The uniqueness of $f$ follows from the fact that $h$ is a monomorphism. \hfill \square

Let $A$ be a normal subalgebra of a $BF$-algebra $X$. Then the map $p : X \to X/A$ defined by $p(x) = [x]_A$ is a homomorphism, which is called the canonical mapping. Note that $\text{Ker}(p) = A$. If $f : X \to Y$ is a homomorphism, where $X$ is a $BF$-algebra and $Y$ is a $BF_2$-algebra, then $\text{Ker}f$ is a normal subalgebra of $X$. Hence $X/\text{Ker}f$ is a $BF$-algebra.

**Theorem 3.4.** ([2]) Let $X$ be a $BF$-algebra and let $Y$ be $BF_2$-algebras. If $f : X \to Y$ is a homomorphism, then $X/\text{Ker}f \cong \text{Im}f$. In particular, if $f$ is surjective, then $X/\text{Ker}f \cong Y$.

**Lemma 3.5.** Let $X$ be a $BF$-algebra and $Y$ be a $BF_2$-algebra. Let $f : X \to Y$ be a homomorphism. If $A$ is a normal subalgebra of $X$ such that $A \subseteq \text{Ker}(f)$, then a map $\bar{f} : X/A \to Y$ defined by $\bar{f}([x]_A) = f(x)$ for any $x \in X$ is a homomorphism.

*Proof.* If $[x]_A = [y]_A$, then $[x * y]_A = [x]_A * [y]_A = [0]_A$. This means $x * y \in A \subseteq \text{Ker}(f)$, and so $f(x) * f(y) = f(x * y) = 0$. By a similar way, $f(y) * f(x) = f(y * x) = 0$. Since $Y$ is a $BF_2$-algebra, we have $f(x) = f(y)$. Therefore $\bar{f}$ is well-defined Clearly, $f$ is a homomorphism. \hfill \square

**Theorem 3.6.** Let $X$ be a $BF$-algebra and let $Y$ be a $BF_2$-algebra. Let $A$ be a normal subalgebra of $X$ and let $f : X \to Y$ be a homomorphism. Then the following are equivalent:

(i) there exists a unique homomorphism $\bar{f} : X/A \to Y$ such that $\bar{f} \circ p = f$, where $p : X \to X/A$ is the canonical mapping.

(ii) $A \subseteq \text{Ker}(f)$.

Furthermore, $\bar{f}$ is a monomorphism if and only if $A = \text{Ker}(f)$.

*Proof.* (i)⇒(ii) If $a \in A$, then $f(a) = \bar{f}(p(a)) = \bar{f}([0]_A) = f(0) = 0$ for all $a \in A$, since $\bar{f} \circ p = f$. Hence $a \in \text{Ker}(f)$.
(ii)⇒(i) By Lemma 3.5, we have a homomorphism \( f : X/A \to Y \) defined by \( f(x) = f(x) \) for all \( x \in X \). Since \( (f \circ p)(x) = f([x]_A) = f(x) \) for all \( x \in X \), we have \( f \circ p = f \). The uniqueness of \( f \) follows from the fact that \( p \) is surjective.

Furthermore, \( f \) is a monomorphism if and only if \( f(x) = 0 \) implies \( [x]_A = 0 \), i.e., \( \text{Ker}(f) \subseteq A \). This proves the theorem. \( \square \)

**Theorem 3.7.** Let \( X \) be a BF-algebra and let \( Y \) be a BF\(_2\)-algebra. Let \( f : X \to Y \) be a homomorphism, and let \( A, B \) be normal subalgebras of \( X \) and \( Y \) respectively such that \( f(A) \subseteq B \). Then there exists a unique homomorphism \( h : X/A \to Y/B \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{q} \\
X/A & \xrightarrow{h} & Y/B
\end{array}
\]

commutes, where \( p \) and \( q \) are canonical epimorphisms.

**Proof.** Define \( h : X/A \to Y/B \) by \( h([x]_A) = [f(x)]_B \). Then \( h \) is well-defined.

In fact, if \( [x]_A = [y]_A \), where \( x, y \in X \), then \( x \cdot y, y \cdot x \in A \) and hence \( f(x) \cdot f(y) = f(x \cdot y) \in f(A) \subseteq B \). By Corollary 2.3, \( f(y) \cdot f(x) \in B \). Since \( Y \) is a BF\(_2\)-algebra, \( [f(x)]_B = [f(y)]_B \) and so \( h \) is well-defined.

Let \( [x]_A, [y]_A \in X/A \). Then \( h([x]_A \cdot [y]_A) = h([x \cdot y]_A) = [f(x \cdot y)]_A = [f(x)]_B \cdot [f(y)]_B = h([x]_A) \cdot h([y]_A) \). Hence \( h \) is a homomorphism.

To prove the commutativity of the diagram, let \( [x]_A \in X/A \). Then \( (h \circ p)(x) = h(p(x)) = h([x]_A) = [f(x)]_B = (q \circ f)(x) \) for all \( x \in X \), i.e., \( h \circ p = q \circ f \).

Finally, to prove the uniqueness of \( h \), let \( k : X/A \to Y/B \) be a homomorphism such that \( k \circ p = q \circ f \). Then \( k([x]_A) = k(p(x)) = (k \circ p)(x) = (q \circ f)(x) = h(p(x)) = h([x]_A) \) for any \( [x]_A \in X/A \), i.e., \( h = k \). This completes the proof. \( \square \)

**Theorem 3.8.** Let \( X \) be a BF-algebra and let \( Y \) be a BF\(_2\)-algebra. If a homomorphism \( f : X \to Y \) can be expressed as a composite of homomorphisms as follows:

\[
X \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} Y,
\]

where \( \alpha \) is an epimorphism, \( \beta \) is an isomorphism, and \( \gamma \) is a monomorphism, then \( A \cong X/\text{Ker}(f) \) and \( B \cong \text{Im}(f) \).

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & X/\text{Ker}f \\
\downarrow{\alpha} & & \downarrow{\kappa} \\
A & \xrightarrow{\beta} & B \\
\downarrow{i} & & \downarrow{\gamma} \\
& \text{Im}(f) & Y
\end{array}
\]

where \(\alpha \) is an epimorphism, \( \beta \) is an isomorphism, and \( \gamma \) is a monomorphism.
where \( i \circ \bar{f} \circ p \) is the canonical decomposition of \( f \) and \( \alpha, \beta, \gamma \) are an epimorphism, an isomorphism and a monomorphism, respectively. Since \( f = \gamma \circ \beta \circ \alpha \) and \( \gamma, \beta \) are each monomorphism, we have \( f(x) = 0 \) if and only if \( \alpha(x) = 0 \). Hence \( \text{Ker}(\alpha) = \text{Ker}(f) = \text{Ker}(p) \). By Theorem 3.2, there exists a unique homomorphism \( h : A \to X/\text{Ker}(f) \) such that \( h \circ \alpha = p \). Clearly the mapping \( h \) is a monomorphism, since \( \text{Ker}(\alpha) = \text{Ker}(p) \). Moreover, \( h \) is surjective, since \( p \) is surjective. Thus \( h \) is an isomorphism.

Since \( \text{Im}(\gamma) = \text{Im}(f) \), by applying Theorem 3.3., we have a unique homomorphism \( k : \text{Im}(f) \to B \) such that \( \gamma \circ k = i \). The mapping \( k \) is clearly an epimorphism. The injectivity of \( k \) follows that \( i \) is injective. Thus \( k \) is an isomorphism.

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**References**


http://dx.doi.org/10.2478/s12175-007-0003-x

Received: January 14, 2015; Published: October 28, 2015