The Least Absolute Deviation Problem
for OWA Operator Weights

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Abstract

We propose a least absolute deviation model for obtaining OWA operator weights:

\[
\text{Minimize} \quad \sum_{i=1}^{n-1} |w_i - w_{i-1}|
\]

subject to \( \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \)

\( w_1 + \cdots + w_n = 1, \quad 0 \leq w_i, i = 1, \cdots, n. \)

Recently, the extended minimax disparity problem was proved by Hong [Fuzzy Sets and Systems, 168 (2011) 35-46]. In this paper, we investigated the equivalence of the solutions for the extended minimax disparity problem and least absolute deviation problem for OWA operator, from theoretical point of view.

Mathematics Subject Classification: 80M50

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1 Introduction

The ordered weighted averaging (OWA) operators were introduced by Yager [11] and have attracted much interest among researchers. An important issue in the theory of OWA operators is the determination of the associated weights. The minimum variance problem was proposed by Fullér and Majlender [4],
which minimizes the variance of OWA operator weights under a given level of orness. Their method requires the solution of the following mathematical programming model:

\[
\text{Minimize} \quad D(W) = \frac{1}{n} \sum_{i=1}^{n} (w_i - \frac{1}{n})^2 \\
\text{subject to} \quad \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\
w_1 + \cdots + w_n = 1, 0 \leq w_i, i = 1, \cdots, n.
\] (1)

Hong [5] gave a new proof of the minimum variance problem. The minimax disparity problem was proposed by Wang and Parkan [9], which minimizes the maximum disparity between two adjacent weights under a given level of orness. This approach was formulated as:

\[
\text{Minimize} \quad \max_{i \in \{1, \cdots, n-1\}} |w_i - w_{i+1}| \\
\text{subject to} \quad \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\
w_1 + \cdots + w_n = 1, 0 \leq w_i, i = 1, \cdots, n.
\] (2)

It is interesting to note that the equivalence of solutions for the minimum variance problem (1) and minimax disparity problems (2) was shown by Liu [7].

The extended minimax disparity problem was proposed by Amin and Emrouznejad [1], which minimizes the maximum disparity of any distinct pairs of weights instead of adjacent weights under a given level of orness. Their method requires the solution of the following nonlinear optimization problem:

\[
\text{Minimize} \quad \max_{i \in \{1, \cdots, n-1\}, j \in \{i+1, \cdots, n\}} |w_i - w_j| \\
\text{subject to} \quad \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\
w_1 + \cdots + w_n = 1, 0 \leq w_i, i = 1, \cdots, n.
\] (3)

This optimization problem was completely proven by Hong [6].

Wang et al. [10] have introduced the following least squares deviation (LSD) method as an alternative approach to determine the OWA operator weights:

\[
\text{Minimize} \quad \sum_{i=1}^{n-1} (w_i - w_{i+1})^2 \\
\text{subject to} \quad \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\
w_1 + \cdots + w_n = 1, 0 \leq w_i, i = 1, \cdots, n.
\] (4)

They solved this model by using LINGO or MATLAB software package. Recently, Song and Liu [8] solved this constrained optimization problem analytically, using the method of Lagrange multipliers. We propose a least absolute
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deviation model (LAD) for determining the OWA operator weights:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{n-1} |w_i - w_{i+1}| \\
\text{subject to} & \quad \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1}w_i = \alpha, \quad 0 \leq \alpha \leq 1, \\
& \quad w_1 + \cdots + w_n = 1, \quad 0 \leq w_i, i = 1, \cdots, n.
\end{align*}
\]

In this paper, we investigated the equivalence of the solutions for the extended minimax disparity problem and least absolute deviation problem for OWA operator, from theoretical point of view.

2 The solution equivalence of the two problem

Liu [7] showed the solution equivalence of minimax disparity and minimum variance problems for OWA operators. Likewise, we proved that the solutions of extended minimax disparity OWA operator problem under given orness level and the least absolute deviation OWA operator problem under given orness level are equivalent.

Recently, Hong [6] proved the extended minimax disparity OWA operator problem (3) suggested by Amin and Emrouznejad [1].

\[H(\alpha) = \text{Minimize}\{\max_{i \in \{1, \cdots, n-1\}, j \in \{i+1, \cdots, n\}} |w_i - w_j|\} = \left|\frac{(1-2\alpha)(n-1)}{(n-m)m}\right|,\]

where \(w_1^* = w_2^* = \cdots = w_m^*, w_{m+1}^* = w_{k+2}^* = \cdots = w_n^*\),

and

\[
w_{m+1}^* = \frac{n - m - (2\alpha - 1)(n-1)}{n(n-m)}.
\]

Here \(m\) satisfies the following: for \(n=2k\)

\[
m = \begin{cases} 
[(1-2\alpha)(n-1)] & \text{if } 0 \leq \alpha \leq (n-2)/4(n-1), \\
\left\lfloor \alpha \right\rfloor & \text{if } (n-2)/4(n-1) \leq \alpha \leq (3n-2)/4(n-1), \\
n - \left\lfloor (2\alpha-1)(n-1) \right\rfloor & \text{if } (3n-2)/4(n-1) \leq \alpha \leq 1,
\end{cases}
\]

where \([x] = m + 1 \iff m < x \leq m + 1\) for any integer \(m\),
and for \(n=2k+1\)
\[ m = \begin{cases} \lceil (1-2\alpha)(n-1) \rceil & \text{if } 0 \leq \alpha \leq 1/4, \\ k \text{ or } k+1, & \text{if } 1/4 \leq \alpha \leq 3/4, \\ n - \lceil (2\alpha - 1)(n-1) \rceil & \text{if } 3/4 \leq \alpha \leq 1. \end{cases} \]

**Theorem 2.2** An optimal weight for the constrained optimization problem (5) is the same as the optimal weight for the constrained optimization problem (3) for a given level of \( \alpha = \text{orness}(W) \).

**Proof.** By Theorem 1, there exist \( W^* = (w^*_1, w^*_2, \ldots, w^*_n) \) such that

\[ w^*_1 = w^*_2 = \cdots = w^*_m, \quad w^*_m+1 = w^*_m+2 = \cdots = w^*_n, \]

and \( H(\alpha) = |w^*_1 - w^*_{m+1}| \). Suppose that

\[ \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1, \]

where \( w_1 + \cdots + w_n = 1, 0 \leq w_i, i = 1, \ldots, n \). Let

\[ \max_{i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}} |w_i - w_j| = |w_{i_0} - w_{j_0}| \]

for some \( i_0, j_0 \) and \( i_0 < j_0 \). Then

\[ \sum_{i=1}^{n-1} |w_i - w_{i+1}| \geq \sum_{i=i_0}^{j_0-1} |w_i - w_{i+1}| \]

\[ \geq |w_{i_0} - w_{j_0}| \]

\[ = \max_{i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}} |w_i - w_j| \]

\[ \geq \min \{ \max_{i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}} |w_i - w_j| \} \]

\[ = |w^*_1 - w^*_{m+1}| \]

\[ = \sum_{i=1}^{n-1} |w^*_i - w^*_{i+1}| \]

where the second inequality comes from triangle inequality. Therefore \( W^* \) is an optimal solution of the constrained optimization problem (5). We now prove the uniqueness. Let \( W' = (w'_1, w'_2, \ldots, w'_n) \) be an optimal solution of the constrained optimization problem (5). Let

\[ \max_{i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}} |w'_i - w'_j| = |w'_{i_1} - w'_{j_1}| \]

for some \( i_1, j_1 \) and \( i_1 < j_1 \). Then

\[ |w^*_1 - w^*_{m+1}| = \sum_{i=1}^{n-1} |w^*_i - w^*_{i+1}| \]
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\[ \geq \sum_{i=1}^{i=n-1} |w_i' - w_{i+1}'| \]

\[ \geq |w_i'_{i1} - w_j'_{j1}| \]

\[ = \max_{i \in \{1, \ldots, n-1\}, j \in \{i+1, \ldots, n\}} |w_i' - w_j'|, \]

that is, \( W' = (w'_1, w'_2, \ldots, w'_n) \) is an optimal solution of the constrained optimization problem (3). Therefore \( W^* = W' \) by Theorem 2.1, which complete the proof.

References


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