Functional Response to Predators Holling type II, as a Function Refuge for Preys in Lotka-Volterra Model

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Abstract

We show what physical capacity of refuge it influences in the existence and stability the unique equilibrium point at interior of the first quadrant. We analyze the consequences of such function through modifying the well-known Lotka-Volterra predator-prey model with prey self-limitation.

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1　Introduction

In population dynamics the models proposed for interactions prey-predator have considered diverse suppositions to simplify their mathematical descriptions, such as: The populations' homogeneity, homogeneity of environmental, distribution spatial uniform, constant rates of growth, encounters between the species predators and equally probable prey, sizes population clerks exclusively of the time, the species predators feeds exclusively of the species prey, while this feeds of a resource that is in the habitat in big quantities the one which alone it intervenes passively, they are not considered behaviors of the species of physiologic, morphological, social type, neither reintroduction of species, etc.

The behavior of the species is affected by the effect of ecological variables (readiness of refuges, formation of defence groups, difficulty of mating, appearance of other strategies antipredator, etc).

The behaviors antipredator can be:

- Physiologic: Emission of chemical substances or pheromones, etc.

- Morphological: Patterns of coloration, mimicry with the atmosphere, adaptability of some parts of the body, etc.

- Imitation of codes: Emission of sounds, to be made the dead, etc.

- Habitat adaptation: Uses of refuges, in the nature, many preys respond to the attacks of the predators looking for such space refuges. The effect itself of refuge use on the population growth is complex in nature, but for modeling purposes it can be understood as the reduction of prey mortality due to reduction in predation success. The refuges affect positively the population growth of preys and negatively that of predators. A more relevant behavior trait that affects the dynamics of predator-prey systems is the use of spatial refuges by the prey. Spatial refuges are found where environmental heterogeneity provides less-accessible sites for predators in which a number of preys can stay, at least temporarily. In this way, some fraction of the prey population is partially protected against predators and we assume that the refuge is a physical location in which prey either live or temporally hide [1]. The majority of the works show the refuge conclusion that stabilizes predator interactions [2]. However, González-Olivares and Ramos-Jiliberto, discard the common conclusion that the use of the shelter by the population of prey always leads to stability as considering the same assumptions in the model Rosenzweig-McArthur obtained which trajectories can oscillate for some parameter values [3].

We denote for $X(t) = X$ and $Y(t) = Y$ the population sizes of preys and predators, respectively for $t \geq 0$, considered as continuous variables that can represent density, biomass or quantity of each population's individuals. The

common Lotka-Volterra predator prey model is

$$X_\varphi : \begin{cases} \frac{dX}{dt} = r(1 - \frac{X}{K})\, X - q\, XY \\[2mm] \frac{dY}{dt} = (b\, X - c)\, Y, \end{cases} \tag{1}$$

where $\varphi = (a,\, b,\, c,\, K,\, r) \in R_+^5$ is vector of biological parameter.
Usually the parameters have the following biological meanings:
r is the intrinsic per capita prey growth rate;
K is the prey environmental carrying capacity;
q is the maximal per capita predator consumption rate;
b is the efficiency with which predators convert consumed prey into new predators;
c is the natural per capita death predator rate.
If $X_r(t) = X_r$, a quantity of prey population that occupies a refuge (heterogeneity of the means), the quantity of preys in refuge, then the quantity of preys that interact with the predators is $X - X_r$. The model (1) is transformed in

$$X_\varphi : \begin{cases} \frac{dX}{dt} = r(1 - \frac{X}{K})\, X - q\, (X - X_r)\, Y \\[2mm] \frac{dY}{dt} = (b\, (X - X_r) - c)\, Y, \end{cases} \tag{2}$$

which is a Kolmogorov type system, where functional response is linear [4]. According to [5, 6] two refuge types have usually been considered in the ecological literature.

(I). Those that protect a constant fraction of the preys, $X_r = \beta X$.

Objection: The fraction of hidden preys is a growing linear function; this implies that the refuge readiness is bigger while bigger it is the population size. Then the system is topologically equivalent to the original one [3], changing only the coordinates of the positive equilibrium point.

(II). Those that protect a fixed quantity of preys, $X_r = \beta$.

Objections: Quantity of refuge prey does not depend on physical capacity of refuge. The occurrence of a constant number or constant proportions of the prey in refuge seem to be very unlikely in the nature [7].

The system ceases to be a Kolmogorov type, but the new positive equilibrium point is also globally asymptotically stable.
In the present work, we analyze the population consequences of refuge use in the Lotka-Volterra model with self-limitation, assuming that the amount

of prey in refuge is given by the function of the functional response type II Holling, which considers the two objections is used

$$X_r = \frac{\alpha X}{X + \beta},$$

where α represents the maximum physical capacity of the refuge and β is a half saturation constant of the refuge. Moreover, we have that the per capita fraction of prey population on refuge $\frac{X_r}{X}$ is a decreasing function. We note that

(i) If $\beta \to 0$ or $X \to \infty$, then $X_r \to \alpha$, therefore all the prey population is in a refuge with capacity α constant.

(ii) If $\beta \to \infty$, then $X_r \to 0$; therefore, the prey in refuge tends to zero

(iii) If $\alpha \to 0$, then $X_r \to 0$; therefore, the amount of prey on refuge tends to zero.

(iv) If $\alpha \to \infty$, then $X_r \to \infty$; therefore, all population can be in refuge.

The study of prey refuge use is essential for conservation of endangered species creating protected areas to preserve them.

2 The model

Considering the function proposed in the model obtains the field vectorial $X_\mu^{\alpha,\beta}$ described by the autonomous system of differential equations:

$$X_\mu^{\alpha,\beta} : \begin{cases} \frac{dX}{dt} = r(1 - \frac{X}{K})\, X - a\,(X - \frac{\alpha X}{X+\beta})Y \\[2mm] \frac{dY}{dt} = \left(b\,(X - \frac{\alpha X}{X+\beta}) - c\right) Y, \end{cases} \tag{3}$$

where $\mu = (a,\, b,\, c,\, K,\, r,\, \alpha,\, \beta) \in R_+^7$ is vector of biological parameters.
Following the methodology used in [8], we make a reparametrization of the vector field $X_\mu^{\alpha,\beta}$ or the system (3) including changes of variables and a time rescaling given by the diffeomorphism $\varphi : (R^+)^2 \times R \to (R^+)^2 \times R$, such that $\varphi(N, P, \tau) = (KN, \frac{r}{a}P, \frac{KN+\beta}{rK}\tau) = (X, Y, t)$, with $\det J\varphi(N, P, \tau) = \frac{KN+\beta}{a} > 0$ and $A = \frac{\alpha}{K}$, $B = \frac{\beta}{K}$, $C = \frac{Kb}{r}$, $D = \frac{c}{r}$. The vector field in the new coordinates is $X_\eta = \varphi \circ X_\mu^{\alpha,\beta}$, and associated second-order differential equations system is the following Kolmogorov type polynomial:

$$X_\eta = \begin{cases} \frac{dN}{d\tau} = N[(1 - N)(N + B) - (N + B - A)P] \\[2mm] \frac{dP}{d\tau} = P\,[C(N + B - A)N - D(N + B)], \end{cases} \tag{4}$$

where $\eta = (A,\ B,\ C,\ D) \in R_+^4$ and the system (4) is topologically equivalent to system (3). The N-nulclines associated to the system (4) they are given for: $N = 0$ and $P = \frac{(1-N)(N+B)}{N+B-A}$, $N > A - B$. The P-nulclines associated to the system (4) they are given for: $P = 0$ and $N = \frac{H + \sqrt[2]{H^2 + 4BCD}}{2C}$. The Jacobian matrix of system (4) is

$$JX_\eta = \begin{bmatrix} JX_\eta^{11} & JX_\eta^{12} \\ JX_\eta^{21} & JX_\eta^{22} \end{bmatrix}$$

where

$$JX_\eta^{11} = -3N^2 + 2(1 - B - P)N + (B - BP + AP),$$
$$JX_\eta^{12} = -N^2 + (A - B)N,\ JX_\eta^{21} = (2CN + (B - A)C - D)P,$$
$$JX_\eta^{22} = CN^2 + (BC - AC - D)N - BD.$$

Considering the isolated means, where there is not reintroduction of preys X such that overcome the prey environmental carrying capacity K.

3   Main Results

For system (4) we have that

Lemma 3.1. *The set* $\Omega = \left\{(N, P) \in (R_0^+)^2 | 0 \le N \le 1, P \ge 0\right\}$ *is an invariant region of vector field.*

Proof.   (a) As system (4) is of Kolmogorov type, then the

   N- axis and P- axis are invariant sets.

   (b) If $N = 1$, we have that $\frac{dN}{d\tau} = - (1 + B - A) P < 0$, and the trajectories point into to region Ω. $(N,\ P) \in [0,\ 1] \times [0,\ \frac{B}{B-A}] \subset \Omega$, if $B > A$. $(N,\ P) \in [A - B,\ 1] \times [0,\ \infty] \subset \Omega$, if $B \le A$. $\square$

Lemma 3.2. *The trajectories are bounded.*

Proof. Using the Poincaré compactification, we obtain the result. $\square$

Lemma 3.3.   (a) *If* $A \ge \frac{(C-D)(1+B)}{C}$, *the system* (4) *has two equilibrium points:*

   (i) $(N_1^*,\ P_1^*) = (0,\ 0)$.
   (ii) $(N_2^*,\ P_2^*) = (1,\ 0)$.

   (b) *If* $A < \frac{(1+B)(C-D)}{C}$, *the system* (4) *has tree equilibrium points:*

(i) $(N_1^*, P_1^*) = (0, 0)$.

(ii) $(N_2^*, P_2^*) = (1, 0)$.

(iii) $(N_3^*, P_3^*) = (\frac{H + \sqrt[2]{H^2 + 4BCD}}{2C}, \frac{HC - (H^2 + H\sqrt[2]{H^2 + 4BCD} + 2BCD)}{2CD})$, where $H = D + (A - B)C$.

If $B > A$, then $N_3^* \to 0$, $P_3^* \to \frac{B}{B-A}$.

If $B \le A$, then $N_3^* \to A - B$, $P_3^* \to \infty$.

Proof. Considering the equations that define X_η, we obtain the result. □

Lemma 3.4. *(i) The singularity $(N_1^*, P_1^*) = (0, 0)$ is saddle point for all parameter value.*

(ii) The singularity $(N_2^*, P_2^*) = (1, 0)$ is globally asymptotically stable, if $A \ge \frac{(C-D)(1+B)}{C}$ and is saddle point, if $A < \frac{(C-D)(1+B)}{C}$.

Proof. Evaluating the Jacobian matrix we have:

(a) $JX\eta(0,0) = \begin{pmatrix} B & 0 \\ 0 & -BD \end{pmatrix}$.

The eigenvalues: $\lambda_1 = B > 0$, $\lambda_2 = -BD < 0$.

(b) $JX_\eta(1,0) = \begin{bmatrix} -(1+B) & A - (1+B) \\ 0 & C(1 + B - A) - D(1 + B) \end{bmatrix}$.

The eigenvalues: $\lambda_1 = -B - 1 < 0$, $\lambda_2 = C(1 + B - A) - D(1 + B)$. We observe that the sign of λ_2 depends on the value of the parameters like it indicates the hypothesis: If $A \ge \frac{(1+B)(C-D)}{C}$, $\lambda_2 < 0$ and $(1,0)$ is globally asymptotically stable by applying the Poincaré Bendixon theorem.

□

For system (4) we have the following results

Theorem 3.5. *The singularity*

$$(N_3^*, P_3^*) = (\frac{H + \sqrt[2]{H^2 + 4BCD}}{2C}, \frac{HC - (H^2 + H\sqrt[2]{H^2 + 4BCD} + 2BCD)}{2CD})$$

is:

(i) *Center, if $A = \frac{(N^* + B)^2}{2N^* + B - 1}$,*

(ii) *Spiral unstable, if $A > \frac{(N^* + B)^2}{2N^* + B - 1}$,*

(iii) *Spiral stable, if $A < \frac{(N^* + B)^2}{2N^* + B - 1}$.*

Proof. Evaluating the Jacobian matrix at

$$(N^*, P^*) = \left(\frac{H + \sqrt[2]{H^2 + 4BCD}}{2C}, \ \frac{HC - (H^2 + H\sqrt[2]{H^2 + 4BCD} + 2BCD)}{2CD} \right),$$

we have:

$$JX_\eta(N^*, P^*) = \begin{bmatrix} JX_\eta^{11} & JX_\eta^{12} \\ JX_\eta^{21} & JX_\eta^{22} \end{bmatrix},$$

where

$$JX_\eta^{11} = -3N^{*2} - 2N^*B + 2N^* - 2N^*P^* + B - P^*B + P^*A,$$
$$JX_\eta^{12} = -(N^* + B - A)N^*, \ JX_\eta^{21} = (2CN^* + CB - CA - D)P^*,$$
$$JX_\eta^{22} = 0.$$

As $P = \frac{(1-N)(N+B)}{N+B-A}$, then $-3N^2 - 2NB + 2N - 2NP + B - PB + PA = -N\left(\frac{N^2 + 2NB - 2NA + B^2 - BA + A}{N+B-A}\right)$, and

$$JX_\eta(N^*, P^*) = \begin{bmatrix} -N^*\left(\frac{N^{*2} + 2N^*B - 2N^*A + B^2 - BA + A}{N^* + B - A}\right) & -(N^* + B - A)N^* \\ (2CN^* + CB - CA - D)P^* & 0 \end{bmatrix},$$

where

$$Det JX_\eta(N^*, P^*) = (N^* + B - A)N^*(2CN^* + CB - CA - D)P* > 0.$$

With $CN^2 + (CB - CA - D)N - DB = 0$ then $CA - CB + D = \frac{CN^2 - DB}{N}$, $(2CN + CB - CA - D)P = \frac{1}{N}(CN^2 + DB)P > 0$ and $N + B - A > 0$ this way $(N^* + B - A)N^*(2CN^* + CB - CA - D)P* = (N^* + B - A)(CN^* + DB)P* > 0$. And the behavior of singularity depends on the trace

$$Trace JX_\eta(N^*, P^*) = -N^*\left(\frac{N^{*2} + 2N^*B - 2N^*A + B^2 - BA + A}{N^* + B - A}\right).$$

If $A = \frac{(N^*+B)^2}{2N^*+B-1}$ the *Trace* is zero and (N^*, P^*) is center. If $A > \frac{(N^*+B)^2}{2N^*+B-1}$ the *Trace* is positive, and (N^*, P^*) is unstable spiral. If $A < \frac{(N^*+B)^2}{2N^*+B-1}$ the *Trace* is negative, and (N^*, P^*) is spiral attractor. □

4 Conclusions

The preys react fleeing the refuge due to the presence of a certain amount of predator. This work was shown in Theorem 3.5, the assertion that regular use refuge stabilizes prey predator-prey interaction depends on the size of the refuge. We show that this way of modeling the refuge function limits cycles are obtained, which indicates periodic solutions. It is important to note that

the parameter of maximum physical capacity of refuge influences in the number of equilibrium points. The variations of the parameter A makes that you of the bifurcation of Hopf and spends from a point of stable equilibrium to a limit cycle. The existence of an equilibrium point within the first quadrant is conditional refuge size, because if $A \rightarrow \frac{(1+B)(C-D)}{C}$, collapses to the point $(N, P) = (1, 0)$. The knowledge of the impact of prey refuge use by a fraction of prey-population is relevant in the context of bioeconomic and conservation management, because it helps in regulating the harvesting activity in the ecosystem and management of reserves or non-take zones; also it is essential for conservation of endangered species creating protected areas (reserves) for preserve them [9]. For this reason, other models for interaction between species must be considered.

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