Abrupt Change of the Mathematical Expectation of the Random Process with Unknown Intensity

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Abstract

In terms of the new approximation of the decision statistics, technically unsophisticated approach is suggested to the determination of the abrupt change of the statistical characteristics of the fast-fluctuating Gaussian processes, the unknown parameters measured before and after jumping. And in terms of the stepwise change of the mathematical expectation of the random process with unknown intensity the corresponding detection and estimation algorithms have been synthesized. The introduced analytical ways are demonstrated for the calculation of the performance of the synthesized detector and measurer for any signal-to-noise ratios. By means of statistical simulation, it is established that the
presented method of determining stepwise change of the statistical characteristics of random processes is efficient indeed. It has also been found out that the theoretical formulas for the characteristics of detection of the stepwise change and estimation of the unknown jumping moment, mathematical expectation and intensity adequately approximate the corresponding experimental data within a wide range of the analyzed process parameters values.

**Keywords**: random process, abrupt change, maximum likelihood method, parametrical prior uncertainty, discontinuous parameter, local Markov approximation method, detection and estimation characteristics, statistical simulation

1 **Introduction**

The problem of detection and measurement of the abrupt change (jumping at some time point) of the mathematical expectation of the random process are studied in a number of works [1-3, etc.]. In certain publications, the statement of this problem is accompanied by the assumption that the observable data realization has a normal distribution. As a rule, the additional restrictions are also imposed referring to the processed samples being uncorrelated [1] or to the model classes of the information signal [2, 3], etc. Besides, in many cases the synthesis of detection and estimation algorithms of the abrupt change is usually conducted in the conditions of complete prior certainty concerning spurious parameters of the analyzed random process.

In the present work, we suggest a technically unsophisticated approach to the determination of the moment of the stepwise change of the mathematical expectation of the band Gaussian random process [4-6] with unknown intensity. The simpler procedure for measuring this process’ power parameters before and after jumping is also considered. Analytically such process can be presented as follows

\[ \xi(t) = \nu(t) + a_{01} + (a_{02} - a_{01}) \theta(t - \lambda_0). \]  

Here \( \theta(t) = 0 \), if \( t < 0 \), and \( \theta(t) = 1 \), if \( t \geq 0 \) – Heaviside function, \( \lambda_0 \) – the moment of possible stepwise change, \( a_{01}, a_{02} \) – mathematical expectations (mean values) of the process \( \xi(t) \) under \( t < \lambda_0 \) and \( t \geq \lambda_0 \), accordingly, and \( \nu(t) \) – stationary centered Gaussian random process, possessing spectral density

\[ G(\omega) = \begin{cases} d_0/2, & |\omega| \leq \Omega/2, \\ 0, & |\omega| > \Omega/2. \end{cases} \]  

In Eq. (2) the designations are: \( \Omega \) – bandwidth, and \( d_0 \) – spectral density magnitude (intensity) of the process \( \nu(t) \) determining its dispersion \( D_0 = d_0 \Omega/4\pi \).
We believe that the process (1) is observed against additive Gaussian white noise \( n(t) \) with one-sided spectral density \( N_0 \). As a result, the mix
\[
x(t) = \xi(t) + n(t), \quad t \in [0, T]
\]
is accessible to be observed. The fluctuations of the process \( \xi(t) \) are now considered as “fast”, so the following condition is satisfied
\[
\mu_{\min} = T_{\min} \Omega / 4\pi \gg 1
\]
where \( T_{\min} = \min(\lambda_0, T - \lambda_0) \). With the observable realization (3), it is necessary to detect the moment of the stepwise change and to estimate the parameters \( \lambda_0, a_{01}, a_{02}, d_0 \) possessing the values within corresponding prior intervals \([\Lambda_1, \Lambda_2], (-\infty, \infty), (-\infty, \infty), [0, \infty)\).

2 Detection of the Stepwise Change of the Mathematical Expectation of the Random Process with Unknown Intensity

For the solution of the problem of detection of the process \( \xi(t) \) mathematical expectation stepwise change, we separate two possible cases (two hypotheses): 1) \( a_{01} = a_{02} \), i.e. jumping is absent (\( H_0 \) hypothesis); 2) \( a_{01} \neq a_{02} \) (\( H_1 \) hypothesis). The problem of the specified hypotheses testing is solved by means of the maximum likelihood method. For this purpose, with the results of the previous studies [5-7] in mind, the expressions for decision statistics (logarithms of the functionals of likelihood ratio) for hypotheses \( H_0, H_1 \) against alternative \( H: x(t) = n(t) \) are written down as
\[
H_0: \quad L_0(a_1, d) = \frac{d}{N_0(N_0 + d)} \int_0^T y^2(t) dt + \frac{2a_1}{N_0 + d} \int_0^T x(t) dt - \frac{a_1^2 T}{N_0 + d} - \frac{\Omega T}{4\pi} \ln \left( 1 + \frac{d}{N_0} \right),
\]
\[
H_1: \quad L_1(\lambda, a_1, a_2, d) = \frac{d}{N_0(N_0 + d)} \int_0^T y^2(t) dt + \frac{2a_1}{N_0 + d} \int_0^T x(t) dt - \frac{a_1^2 T}{N_0 + d} + \frac{2a_2}{N_0 + d} \int_0^\lambda x(t) dt - \frac{a_2^2(T - \lambda)}{N_0 + d} - \frac{\Omega T}{4\pi} \ln \left( 1 + \frac{d}{N_0} \right).
\]
Here \( y(t) = \int_{-\infty}^\infty x(t') h(t - t') dt' \) is the output signal of the filter with the transfer function \( H(\omega) \) satisfying the condition \( |H(\omega)|^2 = 2G(\omega)/d_0 \) (2), and \( \lambda, a_1, a_2, d \) are current values of the parameters \( \lambda_0, a_{01}, a_{02}, d_0 \), accordingly. The choice is made in favor of the stepwise change presence, if [5, 7, 8]...
\[
\max_{\lambda \in [\lambda_1, \lambda_2]} L_4(\lambda, a_1, a_2, d) - \max_{a_1, d \geq 0} L_0(a_1, d) > c, \quad (6)
\]

where \( c \) is the threshold calculated according to the accepted optimality criterion.

Maximization of functionals (5) on variables \( a_1, a_2, d \) can be performed analytically. As a result, it is found that

\[
L_{0 \text{max}} = \max_{a_1, d} L_0(a_1, d) = \mu \left\{ \frac{1}{\mu N_0} \int_0^T y^2(t) dt - \ln \left[ \frac{1}{\mu N_0} \left( \int_0^T y^2(t) dt - \frac{T}{\mu} \left( \int_0^T x(t) dt \right)^2 \right) \right] \right\} - 1,
\]

\[
L_{1 \text{max}}(\lambda) = \max_{a_1, a_2, d} L_4(\lambda, a_1, a_2, d) = L_{0 \text{max}} + \mu \ln \left[ \frac{\int_0^T y^2(t) dt - \frac{T}{\mu} \left( \int_0^T x(t) dt \right)^2}{\int_0^T y^2(t) dt - \frac{1}{\lambda} \left( \int_0^\lambda x(t) dt \right)^2 - \frac{1}{T - \lambda} \left( \int_0^T x(t) dt \right)^2} \right].
\]

Here \( \mu = T \Omega/4\pi \).

From Eqs. (6), (7) follows that maximum likelihood detection algorithm of stepwise change of the mathematical expectation of Gaussian random process with unknown intensity has the form

\[
\max_{\lambda \in [\lambda_1, \lambda_2]} M(\lambda) > c, \quad M(\lambda) = M_1^2(\lambda)/\lambda + M_2^2(\lambda)/(T - \lambda) - M_3^2/T, \quad (8)
\]

\[
M_1(\lambda) = \int_0^\lambda x(t) dt, \quad M_2(\lambda) = \int_\lambda^T x(t) dt, \quad M_3 = \int_0^T x(t) dt,
\]

and it is also an invariant to spectral density of white noise.

In practice, the maximum likelihood detector of the stepwise change of the mathematical expectation of the Gaussian random process with unknown intensity can be implemented in the form of the block diagram selected by the dashed line in Fig. 1. Here the following designations are introduced: 1 is the switch that is open for time \([0, T]\); 2 is an integrator; 3 is a delay line for the period \( T \); 5 is the squarer; 6 is the substracter; 7 is the divider; 8 is the ramp generator; 9 is the adder; 10 is the peak detector; 11 is the threshold device performing comparison between the input signal and the threshold \( c \) and deciding on the presence of stepwise change of the mathematical expectation of the random process within the interval \([0, T]\), in case when the threshold is exceeded, or deciding on the absence of such stepwise change, in case when the threshold is not exceeded.

Detection quality is characterized by type I (false alarm) and type II (missing) error probabilities, designated as \( \alpha \) and \( \beta \), respectively [6, 8]. In order to determine \( \alpha \) and \( \beta \), the normalized functionals

\[
\frac{1}{\lambda} \left( \int_0^\lambda x(t) dt \right)^2 - \frac{1}{T - \lambda} \left( \int_0^T x(t) dt \right)^2.
\]
Abnormal change of the mathematical expectation of the random process

\[ \hat{M}_i(l) = M_i(\lambda_0) \sqrt{\frac{2}{T(N_0 + d_0)}}, \quad i = 1, 2, \quad \hat{M}_3 = M_3 \sqrt{\frac{2}{T(N_0 + d_0)}} \]  

are considered, presented as the sums of signal and noise functions [8]:

\[ \hat{M}_i(l) = S_i(l) + N_i(l), \quad \hat{M}_3 = S_3 + N_3. \]  

Here \( S_i(l) = \langle \hat{M}_i(l) \rangle \), \( S_3 = \langle \hat{M}_3 \rangle \) are signal, \( N_i(l) = \hat{M}_i(l) - \langle \hat{M}_i(l) \rangle \), \( N_3 = \hat{M}_3 - \langle \hat{M}_3 \rangle \) are noise functions, \( l = \lambda / T \) is current value of the normalized parameter \( l_0 = \lambda_0 / T \), and the averaging \( \langle \cdot \rangle \) is performed in terms of the all possible realizations \( x(t) \) with fixed values for the all unknown parameters \( \lambda_0, a_{01}, a_{02}, d_0 \). While executing the ratio (4), for signal functions and correlation functions of noise functions specified above we have

\[
S_1(l) = z_{01}l + \Delta \varepsilon \max(0, l - l_0), \quad S_2(l) = z_{01}(1 - l) + \Delta \varepsilon \left[ 1 - \max(l, l_0) \right], \\
S_3 = z_{01} + \Delta \varepsilon (1 - l_0), \\
\langle N_1(l_1)N_1(l_2) \rangle = \min(l_1, l_2), \quad \langle N_2(l_1)N_2(l_2) \rangle = 1 - \max(l_1, l_2), \quad \langle N_3^2 \rangle = 1
\]  

Fig. 1. Block diagram of the detector/measurer of stepwise change of the mathematical expectation of the random process with unknown intensity

Here \( S_i(l) = \langle \hat{M}_i(l) \rangle \), \( S_3 = \langle \hat{M}_3 \rangle \) are signal, \( N_i(l) = \hat{M}_i(l) - \langle \hat{M}_i(l) \rangle \), \( N_3 = \hat{M}_3 - \langle \hat{M}_3 \rangle \) are noise functions, \( l = \lambda / T \) is current value of the normalized parameter \( l_0 = \lambda_0 / T \), and the averaging \( \langle \cdot \rangle \) is performed in terms of the all possible realizations \( x(t) \) with fixed values for the all unknown parameters \( \lambda_0, a_{01}, a_{02}, d_0 \). While executing the ratio (4), for signal functions and correlation functions of noise functions specified above we have

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S_3 = z_{01} + \Delta \varepsilon (1 - l_0), \\
\langle N_1(l_1)N_1(l_2) \rangle = \min(l_1, l_2), \quad \langle N_2(l_1)N_2(l_2) \rangle = 1 - \max(l_1, l_2), \quad \langle N_3^2 \rangle = 1
\]  

Fig. 1. Block diagram of the detector/measurer of stepwise change of the mathematical expectation of the random process with unknown intensity
Here $\Delta z = z_{02} - z_{01}$, and $z_{0i} = a_{0i} \sqrt{2T/(N_0 + d_0)}$, $i = 1, 2$ are signal-to-noise ratios (SNR) for constant components $a_{0i}$ within the whole observation interval.

With Eq. (9) in mind, we now rewrite the decision rule in the form of

$$\max_{l \in [\bar{\lambda}_1, \bar{\lambda}_2]} \tilde{M}(l) > u.$$ \hspace{1cm} (12)

where $\tilde{M}(l) = 2M(\lambda)/T(N_0 + d_0) = \tilde{M}_1^2(l)/l + \tilde{M}_2^2(l)/(1-l) - \tilde{M}_3^2$, $\bar{\lambda}_{1,2} = \Lambda_{1,2}/T$, and $u = 2c/T(N_0 + d_0)$ is the expression of the normalized threshold.

First, let us assume that stepwise change of the mathematical expectation of the process $\xi(t)$ (1) is absent, i.e.

$$z_{01} = z_{02} \quad (\Delta z = 0).$$ \hspace{1cm} (13)

In this case for false-alarm probability $\alpha$ we have

$$\alpha = P[\max_{l \in [\bar{\lambda}_1, \bar{\lambda}_2]} \tilde{M}(l) > u] = 1 - P_N(u),$$ \hspace{1cm} (14)

where $P_N(u) = P[\tilde{M}(l) < u]$, $l \in [\bar{\lambda}_1, \bar{\lambda}_2]$.

Using Eqs. (10)-(13), we present the functional $\tilde{M}(l)$ like that

$$\tilde{M}(l) = \left[N_1(l)/(1-l)/l - N_2(l)/(1-l) + \bar{\lambda}_1/l - \bar{\lambda}_2\right]^2.$$ \hspace{1cm} (15)

Further, in Eq. (15) let us execute the change of variables:

$$0 = \ln[l/(1-l)], \quad 0 \in [\Theta_1, \Theta_2], \quad \Theta_i = \ln[\bar{\lambda}_i/(1-\bar{\lambda}_i)], \quad i = 1, 2.$$ \hspace{1cm} (16)

Then, the probability $P_N(u)$ (14) can be defined as

$$P_N(u) = P[X^2(\Theta) < u], \quad 0 \in [\Theta_1, \Theta_2],$$ \hspace{1cm} (17)

where $X(\Theta)$ is Gaussian random process with zero mathematical expectation and correlation function $\langle X(\Theta_1)X(\Theta_2) \rangle = \exp(-|\Theta_2 - \Theta_1|/2)$. Consequently, with the results from [9] we find

$$P_N(u) = \begin{cases} \exp\left[-(\Theta_2 - \Theta_1)\sqrt{u}/2\pi \exp(-u/2)\right], & u \geq 1, \\ 0, & u < 1, \end{cases}$$ \hspace{1cm} (18)

and the expression for false-alarm probability (14) gets the form

$$\alpha = \begin{cases} 1 - \exp\left[-(\Theta_2 - \Theta_1)\sqrt{u}/2\pi \exp(-u/2)\right], & u \geq 1, \\ 1, & u < 1. \end{cases}$$ \hspace{1cm} (19)

Accuracy of the formula (19) increases with $u$ and ratio

$$m = \bar{\lambda}_2/(1-\bar{\lambda}_1)/\bar{\lambda}_1/(1-\bar{\lambda}_2).$$ \hspace{1cm} (20)
Abrupt change of the mathematical expectation of the random process

Now let us assume that \( z_{01} \neq z_{02} \) and then the stepwise change missing probability is written down as

\[
\beta = P\left\{ \max_{l \in [\tilde{\lambda}_1, \tilde{\lambda}_2]} \tilde{M}(l) < u \right\} = P\left\{ \tilde{M}(l) < u \right\}. \tag{21}
\]

In Eq. (21) we move to the new variable \( y = l/(1-l) \) defined within the interval \( [Y_1, Y_2] \), \( Y_i = \tilde{\lambda}_i/(1-\tilde{\lambda}_i) \), \( i = 1, 2 \), and, with reference to Eq. (11), we present the signal function \( S(y) = \langle \tilde{M}(y) \rangle \) and correlation function \( \langle N(y_1)N(y_2) \rangle \) of the noise function \( N(y) = M(y) - \langle M(y) \rangle \) of the normalized functional \( \tilde{M}(y) \) (12) in the form of

\[
S(y) = \Delta z^2 \left\{ \begin{array}{ll}
(1-l_0)^2 y, & y \leq y_0, \\
\frac{l_0^2}{y}, & y > y_0,
\end{array} \right. \tag{22}
\]

\[
\langle N(y_1)N(y_2) \rangle = 4\Delta z^2 \left\{ \begin{array}{ll}
(1-l_0)^2 \min(y_1, y_2), & y_1, y_2 \leq y_0, \\
\frac{l_0^2}{y_1} \min(l/y_1, l/y_2), & y_1, y_2 > y_0,
\end{array} \right.
\]

where \( y_0 = l_0/(1-l_0) \).

From Eq. (22) follows that within the intervals \( [Y_1, y_0] \) and \( (y_0, Y_2] \) the functional \( \tilde{M}(y) \) is Gaussian Markov diffusion process with drift \( \tilde{K}_1 \) and diffusion \( \tilde{K}_2 \) coefficients taking the form

\[
\tilde{K}_1 = \Delta z^2 \left\{ \begin{array}{ll}
(1-l_0)^2, & y \leq y_0, \\
-\frac{l_0^2}{y^2}, & y > y_0,
\end{array} \right. \quad \tilde{K}_2 = 4\Delta z^2 \left\{ \begin{array}{ll}
(1-l_0)^2, & y \leq y_0, \\
\frac{l_0^2}{y^2}, & y > y_0.
\end{array} \right. \tag{23}
\]

Let us consider that

\[
|\Delta z| >> 1. \tag{24}
\]

Then maximum position of the functional \( \tilde{M}(y) \) is located in the near neighborhood of the point \( y = y_0 \). The variable \( \varepsilon = y - y_0 \) is introduced, the absolute value of which decreases with increasing \( |\Delta z| \). The drift and diffusion coefficients (23) are then rewritten as follows

\[
\tilde{K}_1 = \Delta z^2 \left\{ \begin{array}{ll}
(1-l_0)^2, & y \leq y_0, \\
-\frac{l_0^2}{(y_0 + \varepsilon)^2}, & y > y_0,
\end{array} \right. \quad \tilde{K}_2 = 4\Delta z^2 \left\{ \begin{array}{ll}
(1-l_0)^2, & l \leq l_0, \\
\frac{l_0^2}{(y_0 + \varepsilon)^2}, & l > l_0.
\end{array} \right.
\]

As \( \varepsilon \rightarrow 0 \) if \( |\Delta z| \rightarrow \infty \), then for greater values of \( |\Delta z| \) the functional \( \tilde{M}(y) \) can be approximated in the neighborhood of point \( y = y_0 \) by Gaussian Markov process with drift and diffusion coefficients

\[
\tilde{K}_1 = \Delta z^2 (1-l_0)^2 \left\{ \begin{array}{ll}
1, & y \leq y_0, \\
-1, & y > y_0,
\end{array} \right. \quad \tilde{K}_2 = 4\Delta z^2 (1-l_0)^2.
\]
We use this approximation over all the interval of the possible values of the variable \( y \in [y_1, y_2] \). Then, from the results of the study [10], for the probability (21) we find

\[
\beta = P_S(u),
\]

\[
P_S(u) = \frac{1}{\sqrt{2\pi} \sqrt{K_2(y_0 - y_1) + \sigma_1^2}} \int_0^\infty dx \left\{ \exp \left[ \frac{(u - S_1 - |K_1|(y_0 - y_1) - x)^2}{2(K_2(y_0 - y_1) + \sigma_1^2)} \right] \times \right. \\
\left. \times \Phi \left[ \frac{(y_0 - Y_1)}{\sigma_1 \sqrt{K_2(y_0 - y_1) + \sigma_1^2}} \right] \exp \left( - \frac{|K_1|}{K_2} \left( \frac{y_0 - y_1}{K_2} - \frac{x}{K_2(y_0 - y_1) + \sigma_1^2} \right) \right) \Phi \left( \frac{|K_1|}{K_2} \left( \frac{y_0 - y_1}{K_2} - \frac{x}{K_2(y_0 - y_1) + \sigma_1^2} \right) \right) \right\}.
\]

Here the designations are: \( S_1 = \tilde{S}(Y_1) = \tilde{\Lambda}_1 \Delta z^2 (1 - l_0)^2 / (1 - \tilde{\Lambda}_1) \), \( \sigma_1^2 = \langle \tilde{N}^2(Y_1) \rangle = 4 \tilde{\Lambda}_1 \Delta z^2 (1 - l_0)^2 / (1 - \tilde{\Lambda}_1) \), and \( \Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^x \exp(-t^2/2) \, dt \) is probability integral.

In case when

\[
\tilde{\Lambda}_1 = \epsilon_1, \quad \tilde{\Lambda}_2 = 1 - \epsilon_2,
\]

where \( \epsilon_1, \epsilon_2 << 1 \), the expression (26) is considerably simplified and assumes the form of

\[
P_S(u) = \Phi \left( \frac{u - |K_1| y_0}{\sqrt{K_2 y_0}} \right) \times \left\{ \exp \left[ \frac{2|K_1|}{K_2} \left( \frac{2|K_1| y_0 - u}{K_2} \right) \right] \exp \left[ \frac{4|K_1|}{K_2} \left( \frac{|K_1| y_0 + u}{K_2} \right) \right] \right\}.
\]

Studies has shown that calculations of the missing probability \( \beta \) (21), performed in accordance with the formulas (26) and (28), satisfactorily agree with each other for practically any values of \( \tilde{\Lambda}_1, \tilde{\Lambda}_2 \). This can be explained by the fact that, under the fulfillment of the condition (24), the missing probability (21) is defined by the behavior of the functional \( \widetilde{M}(y) \) (12) in the near neighborhood of the point \( y_0 \) [5]. The form of the approximation of the \( \widetilde{M}(y) \) (12) decision
Abrupt change of the mathematical expectation of the random process is not important and is only defined in the limiting case \(|\Delta z| \to \infty\) by physical validity and consistency of \(\tilde{M}(y)\) characteristics concerning their physical meaning \([5]\).

Accuracy of the formulas (25), (26), (28) increases with \(\Delta z\) and \(u\).

### 3 Estimation of the Stepwise Change Point and Power Parameters of the Random Process

Let us suppose now that stepwise change of the mathematical expectation of random process \(\xi(t)\) (1) is realized with probability 1 within the interval \([0, T]\). And it is necessary to measure the change-point time \(\lambda_0\) jointly with the values of power parameters \(a_{01}, a_{02}, d_0\). The synthesis of the joint estimation algorithm is to be conducted by means of maximum likelihood method. Using Eq. (5), for maximum likelihood estimations (MLEs) \(\lambda_m, a_{1m}, a_{2m}, d_m\) of unknown parameters \(\lambda_0, a_{01}, a_{02}, d_0\) we obtain

\[
\lambda_m = \arg \max_{\lambda \in [\lambda_1, \lambda_2]} M(\lambda),
\]

\[
a_{1m} = M_1(\lambda_m)/\lambda_m, \quad a_{2m} = M_2(\lambda_m)/(T - \lambda_m),
\]

\[
d_m = \max \left\{ 0, \frac{1}{M_1} \int_0^T y^2(t) dt - \frac{1}{\lambda_m} M_1^2(\lambda_m) - \frac{1}{T - \lambda_m} M_2^2(\lambda_m) - N_0 \right\},
\]

where \(\mu\) and \(M(\lambda), M_i(\lambda), i = 1, 2\) are defined from Eqs. (7) and (8), correspondingly.

The measurer (29) can be implemented in the form of the block diagram shown in Fig. 1, from which the block train 5-7-6-10-11 should be excluded. Other designations are the following: 4 is a filter with transfer function \(H(\omega)\) (6), 12 is the retriever of the position of the input signal greatest maximum (extremum), 13 is the sampling device forming to its output the input signal sample at the instant time determined by the value \(\lambda_m\), 14 is the nonlinear element with characteristic \(f(x) = \max(0, x)\).

Let us determine the characteristics of the estimates (29). Using Eqs. (10) and (11), the signal \(S(l)\) and noise \(N(l)\) functions of the normalized functional \(\tilde{M}(l)\) (12) are now presented in the form of

\[
S(l) = \left[ z_{01} l + \Delta z \max(0, l - l_0) \right]^2 / l + \left[ z_{01}(1 - l) + \Delta z (1 - \max(l, l_0)) \right]^2 / (1 - l) - \left[ z_{01} + \Delta z (1 - l_0) \right]^2,
\]

\(30\)
\[ N(l) = 2[z_0 l + \Delta z \max(0, l - l_0)]N_1(l)/l + 2[z_0 l(1-l) + \\
+ \Delta z(1 - \max(l, l_0))]N_2(l)/(1-l) - 2[z_0 + \Delta z(1-l_0)]N_3 + N_2^2(l). \]

Here \( N_2^2(l) = N_1^2(l)/l + N_2^2(l)/(1-l) - N_3^2. \)

It is presupposed that the condition (24) is satisfied, so the last summand in the second expression (30) can be neglected. Thus, the normalized estimate \( l_m = \lambda_m/T \) converges to the true value of the normalized estimated parameter \( l_0 \) in mean square with increasing \( |\Delta z| \) [8]. As a result, under \( |\Delta z| >> 1 \), it is enough to study the behavior of the functional \( \tilde{M}(l) \) in a near neighborhood of the point \( l = l_0 \) in order to define the characteristics of the estimate \( l_m \). The designation introduced is \( \Delta = \max\{l_1 - l_0, |l_2 - l_0|, |l_1 - l_2|\} \). Then, if \( \Delta \to 0 \), for the signal function and correlation function of the noise function (30), the following asymptotic representations are valid:

\[
S(l) \approx \Delta^2 [l_0(1-l_0) - |l - l_0|],
\]

(31)

\[
\langle N(l)N(l') \rangle \approx 4\Delta^2 \left\{ l_0(1-l_0) - |l_1 - l_2| - \min(|l_1 - l_0|, |l_2 - l_0|), (l_1 - l_0)(l_1 - l_0) \geq 0, \right. \\
\left. l_0(1-l_0) - |l_1 - l_2|, \quad (l_1 - l_0)(l_1 - l_0) < 0. \right.
\]

We then introduce the differential functional

\[ \zeta(l) = \tilde{M}(l) - \tilde{M}(x), \quad l, x \in \tilde{A}_\delta. \]

Here \( \tilde{A}_\delta = [l_0 - m_r, l_0 + m_r] \), and the value of \( m_r \) is fixed and it is chosen so small that, under \( \Delta < m_r \), the approximations (31) demonstrate the required accuracy. Then, if \( |\Delta z| >> 1 \) (24), the conditional distribution function \( F_0(x|l_0) \) of the estimate \( l_m \) can be presented in the form of

\[ F_0(x|l_0) = P[l_m < x] = P \left[ \max_{l \leq x} \zeta(l) > \max_{l \geq x} \zeta(l) \right], \quad l, x \in \tilde{A}_\delta. \]

Using the Doob’s theorem in the wording [11], it is easy to show that within the interval \( \tilde{A}_\delta \) the process \( \zeta(l) \) is asymptotically (under \( \mu_{\min} \to \infty \)) Gaussian Markov random process of diffusion type, for which under \( l \geq x \) the drift \( K_1 \) and diffusion \( K_2 \) coefficients are defined by the following expressions

\[ K_1 = \Delta z^2 \left\{ \begin{array}{ll} 1, & l \leq l_0, \\
-1, & l > l_0, \end{array} \right. \quad K_2 = 4\Delta z^2. \]

Then, according to the study [12], for conditional probability density \( w_0(l|l_0) \), bias \( b_0(l_0|l_0) = \langle l_m - l_0 \rangle \) and variance \( V_0(l_0|l_0) = \langle (l_m - l_0)^2 \rangle \) of the estimate \( l_m \) (29) we obtain

\[ \]
\[
w_0(l|\theta_0) = \frac{2K_1^2}{K_2} \left\{ 3 \exp \left( \frac{4K_1^2(l-l_0)}{K_2} \right) \left[ 1 - \Phi \left( \frac{3K_1\sqrt{l-l_0}}{K_2} \right) \right] + \Phi \left( \frac{K_1\sqrt{l-l_0}}{K_2} \right) - 1 \right\} = \\
= \frac{\Delta z^2}{2} \left\{ 3 \exp \left( \Delta z^2(l-l_0) \right) \left[ 1 - \Phi \left( \frac{3\Delta z}{2}\sqrt{l-l_0} \right) \right] + \Phi \left( \frac{\Delta z}{2}\sqrt{l-l_0} \right) - 1 \right\},
\]

(32)

The accuracy of the formulas (32) increases with \( \mu_{\min} \) and \( |\Delta z| \).

In the case of the small values \( |\Delta z| \) (\( |\Delta z| \to 0 \)), the normalized decision statistics (12) can be approximated (the values of the \( |\Delta z| \) order and less) presented in the form of (15), or in some equivalent form

\[
\hat{M}(l) = \left[ N(l) - N_3 \right]^2 / l(1-l).
\]

(33)

In Eq. (33) we make the change of variables (16) and move from the normalized estimate \( l_m \) (29) to the estimate

\[
\theta_m = \arg \max_{\theta \in [\Theta_1, \Theta_2]} \chi^2(\theta).
\]

(34)

Here \( X(\theta) \) is defined in the same way as in Eq. (17).

According to [8], the position of the maximum of the stationary random process is described by the uniform probability density. Then, for conditional probability density \( w(\theta|\theta_0) \) of the random variable \( \theta_m \) (34), where \( \theta_0 = \ln[l_0/(1-l_0)] \), we can write down:

\[
w(\theta|\theta_0) = 1/(\Theta_2 - \Theta_1), \quad \theta \in [\Theta_1, \Theta_2].
\]

Hence, taking into account Eq. (16), under small \( |\Delta z| \) and for the conditional probability density \( w_a(l|\theta_0) \), bias \( b_a(l_m|\theta_0) \) and variance \( V_a(l_m|\theta_0) \) of the MLE \( l_m \) (29), we find

\[
w_a(l|\theta_0) = 1/l(1-l)\ln m, \quad l \in [\tilde{\lambda}_1, \tilde{\lambda}_2],
\]

(35)

\[
b_a(l_m|\theta_0) = 1 - l_0 - \ln(\tilde{\lambda}_2/\tilde{\lambda}_1)/\ln m,
\]

\[
V_a(l_m|\theta_0) = l_0^2 + \left[ 1 - 2l_0 \right] \ln \left( \left( 1 - \tilde{\lambda}_1 \right)/\left( 1 - \tilde{\lambda}_2 \right) \right) + \tilde{\lambda}_1 - \tilde{\lambda}_2 \right]/\ln m,
\]

where \( \tilde{\lambda}_1, \tilde{\lambda}_2 \) and \( m \) are defined according to Eqs. (12), (20).

For arbitrary values of \( |\Delta z| \), the distribution of the estimate \( l_m \) is found in the form of

\[
w(l|\theta_0) = P_0 w_0(l|\theta_0) + (1 - P_0) w_a(l|\theta_0),
\]

(36)

where \( w_0(l|\theta_0), w_a(l|\theta_0) \) are defined from Eqs. (32), (35), \( P_0 = P[H_S > H_N] \).
and $H_S$, $H_N$ are the random variables corresponding to the maxima of the functional $\tilde{M}(l)$ when maximum position (MLE $l_m$) is subject to the distribution law $w_0(\eta_0^p)$ (32) or $w_1(\eta_0^p)$ (35), accordingly.

The probability $P_0$ can be found using two-dimensional probability density $w_2(u,v)$, or distribution function

$$F_2(u,v) = P[H_S < u, H_N < v] = \int_0^u \int_0^v w_2(u',v') du' dv'$$

of the random variables $H_S$, $H_N$ as

$$P_0 = \int_0^\infty \int_0^\infty w_2(u,v) dv du = \int_0^\infty \left[ \frac{\partial F_2(u,v)}{\partial u} \right]_{v=0}^\infty du.$$  \hspace{1cm} (37)

Here it has been taken into account that $H_S > 0$, $H_N > 0$.

As follows from Eqs. (15) and (30), the random variables $H_S$, $H_N$ are uncorrelated. Then Eq. (37) can be approximately presented in the form of

$$P_0 \approx \int_0^\infty P_N(u) dP_S(u),$$  \hspace{1cm} (38)

where $P_N(u)$, $P_S(u)$ are determined according to Eqs. (18), (26). Substituting Eqs. (18), (26) into Eq. (38), we obtain

$$P_0 = \frac{1}{\sqrt{2\pi\tilde{K}_2(y_0-y_1)+\sigma_1^2}} \int_0^\infty dudx \exp \left[ -(\Theta_2-\Theta_1)\sqrt{\frac{u}{2\pi}} \exp \left( -\frac{u}{2} \right) \right] \times$$

$$\times \left( u-S_1 - \tilde{K}_1(y_0-y_1)-x \right) \Phi \left[ \frac{(y_0-y_1)(\tilde{K}_1\sigma_1^2 + \tilde{K}_2(u-S_1)) - \sigma_1^2 x}{\sigma_1 \sqrt{\tilde{K}_2(y_0-y_1)+\sigma_1^2}} \right]$$

$$\times \exp \left( \frac{(u-S_1 - \tilde{K}_1(y_0-y_1)-x)^2}{2(\tilde{K}_2(y_0-y_1)+\sigma_1^2)} \right) \Phi \left[ \frac{(y_0-y_1)(\tilde{K}_1\sigma_1^2 + \tilde{K}_2(u-S_1)) + \sigma_1^2 x}{\sigma_1 \sqrt{\tilde{K}_2(y_0-y_1)+\sigma_1^2}} \right] \times$$

$$\times \left( \Phi \left( \frac{\sqrt{y_2-y_0}}{\tilde{K}_2} + \frac{x}{\sqrt{\tilde{K}_2(y_2-y_0)}} \right) - \exp \left( -\frac{2\tilde{K}_1}{\tilde{K}_2} \right) \Phi \left( \frac{\sqrt{y_2-y_0}}{\tilde{K}_2} - \frac{x}{\sqrt{\tilde{K}_2(y_2-y_0)}} \right) \right).$$  \hspace{1cm} (39)

In the special case (27) the formula (39) is considerably simplified and has the form
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\[ P_0 = \frac{2|\tilde{K}_1|}{K_2} \int_1 \exp \left[ \frac{2|\tilde{K}_1|u}{K_2} - (\Theta_2 - \Theta_1)\sqrt{\frac{u}{2\pi}} \right] \exp \left( -\frac{u}{2} \right) \frac{2\exp \left[ \frac{2|\tilde{K}_1|(|\tilde{K}_1|y_0 + u)}{K_2} \right]}{1 - \Phi \left( \frac{u + 3|\tilde{K}_1|y_0}{\sqrt{K_2y_0}} \right)} + \exp \left[ \frac{4|\tilde{K}_1|(|\tilde{K}_1|y_0 - u)}{K_2} \right] \Phi \left( \frac{u - 3|\tilde{K}_1|y_0}{\sqrt{K_2y_0}} \right) + \Phi \left( \frac{u + |\tilde{K}_1|y_0}{\sqrt{K_2y_0}} \right) \right] du. \]  

(40)

Using Eqs. (36), (39), under arbitrary values \(|\Delta z|\) for conditional bias \(b(l_m|l_0)\) and variance \(V(l_m|l_0)\) of the estimate \(l_m\), we get

\[ b(l_m|l_0) = b_0(l_m|l_0) + (1 - P_0)b_a(l_m|l_0) = (1 - P_0)b_a(l_m|l_0), \]  

(41)

where \(b_0(l_m|l_0), V_0(l_m|l_0), b_a(l_m|l_0), V_a(l_m|l_0)\) are determined from Eqs. (32), (35). It should be noted that the values of the variance \(V(l_m|l_0)\) (21) calculated by means of the formulas (39), (40) and (41) agree with each other well, if \(|\tilde{\Lambda}_i - l_0| > 0.1, i = 1, 2\). The accuracy of the formulas (39)-(41) increases with \(\mu_{min}\) (4), \(m\) (20) and \(|\Delta z|\) (24).

Now let us determine the characteristics of estimates \(a_{1m}, a_{2m}\) and \(d_m\) (29). In the study [13] it is shown that the accuracy of the MLEs of regular parameters (mathematical expectation and spectral density magnitude) does not asymptotically (with increasing SNR) depend on the presence of unknown discontinuous parameter (moment of stepwise change). It means that in the case of greater values of \(|\Delta z|\) (11) the conditional biases and variances of the MLEs (29) coincide asymptotically with the conditional biases and variances of the estimates of the mathematical expectation, and spectral density magnitude of the random process \(\xi(t)\) with a priori known moment of stepwise change. Then, supposing that \(l_m = l_0\) in Eq. (29), directly averaging over all the possible realizations of the observable data (3) at the fixed values \(a_{01}, a_{02}, d_0\) and taking into account Eq. (4) for the characteristics of the estimates (29), we now find

\[ b_0(a_{1m}|a_{01}) = \langle a_{1m} - a_{01} \rangle = 0, \quad V_0(a_{1m}|a_{01}) = \left\langle (a_{1m} - a_{01})^2 \right\rangle = (N_0 + d_0)/2\lambda_0, \]

\[ b_0(a_{2m}|a_{02}) = \langle a_{2m} - a_{02} \rangle = 0, \quad V_0(a_{2m}|a_{02}) = \left\langle (a_{2m} - a_{02})^2 \right\rangle = (N_0 + d_0)/2(T - \lambda_0), \]

\[ b(d_m|d_0) = \langle d_m - d_0 \rangle = -2\pi(N_0 + d_0)/T\Omega, \]

(42)

\[ V(d_m|d_0) = \left\langle (d_m - d_0)^2 \right\rangle = 4\pi(N_0 + d_0)^2/T\Omega. \]

The accuracy of the formulas (42) increases with \(\mu_{min}\) and \(|\Delta z|\).
4 Results of the Statistical Simulation

In order to establish the borders of applicability for the found asymptotically exact formulas for detection and estimation characteristics, we demonstrate the statistical computer simulation of the algorithms (8), (29), using a procedure presented in [12]. During simulation within the interval \([\bar{\lambda}_1, \bar{\lambda}_2]\) (12) with step \(\Delta l = 10^{-5}\), the samples of realizations of the normalized functionals \(\tilde{M}_1(l), \tilde{M}_2(l)\) (9), \(\tilde{M}(l)\) (12) are generated and the value of the normalized random variable \(\tilde{M}_x = (1/\mu N_0)\int_0^T y^2(t) dt\) (29) is also formed in both presence and absence of the stepwise change. For each realization of \(x(t)\) (3), the maximal sample (12) is compared with the threshold \(u\) and the false-alarm (if \(a_{01} = a_{02}\)), and the missing (if \(a_{01} \neq a_{02}\)) probabilities are found. Besides, according to Eq. (29), the estimates \(\lambda_m, a_{1m}, a_{2m}, d_m\) are defined, and their variances are calculated.

Some results of the statistical simulation are presented in Figs. 2-7 where corresponding theoretical dependences are also shown. Each experimental value is obtained by processing no less than \(5 \cdot 10^4\) realizations of \(x(t)\). In Fig. 2 theoretical dependence (19) of the false-alarm probability \(\alpha(u)\) is drawn. Borders of the prior interval of possible values of parameter \(l_0\) (11) are assumed equal to \(\bar{\lambda}_1 = 0.1, \bar{\lambda}_2 = 0.9\). Experimental values of the false-alarm probability (14) for \(\bar{\lambda}_1 = 0.1, \bar{\lambda}_2 = 0.9\) and \(\mu = 200, 500, 1000\) are designated by squares, crosses, rhombuses.

In Fig. 3 the theoretical dependences of the missing probability \(\beta(\Delta z)\) (25), (26) are represented. The curve 1 is calculated for the case when \(l_0 = 0.05\), \(\bar{\lambda}_1 = 0.025, \bar{\lambda}_2 = 0.975\), \(2 - l_0 = 0.125\), \(\bar{\lambda}_1 = 0.1, \bar{\lambda}_2 = 0.9\), \(3 - l_0 = 0.5, \bar{\lambda}_1 = 0.1, \bar{\lambda}_2 = 0.9\). The threshold \(u\) is determined from Eq. (19) by Neumann-Pirson criterion according to the set level of the false-alarm probability equal to 0.01. Corresponding experimental values of the missing probability \(\beta\) are designated by squares, crosses, rhombuses.

In Fig. 4, by solid lines the dependences of the conditional variance \(V_f(\Delta z) = V(l_m|l_0)\) (39), (41) are plotted, and by dash line – the conditional variance \(V_{0f}(\Delta z) = V_0(l_m|l_0)\) (32) of the normalized estimate of the moment of stepwise change \(l_m\) (29). The curve 1 is calculated for \(l_0 = 0.05, \bar{\lambda}_1 = 0.04, \bar{\lambda}_2 = 0.96, 2 - l_0 = 0.125, \bar{\lambda}_1 = 0.1, \bar{\lambda}_2 = 0.9\), \(3 - l_0 = 0.25, \bar{\lambda}_1 = 0.1, \bar{\lambda}_2 = 0.9\). Experimental values of the variance of the estimate \(l_m\) corresponding to the curves 1-3 are designated by squares, crosses, rhombuses.
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In Figs. 5, 6 the theoretical dependences (42) of the normalized conditional variances \( V_{z1} = 2TV(a_{lm}|a_0)/(N_0 + d_0) \), \( V_{z2} = 2TV(a_{2m}|a_0)/(N_0 + d_0) \) of the estimates \( a_{lm}, a_{2m} \) (29) are shown. Curves 1 correspond to \( l_0 = 0.125, 2 - 0.25, 3 - 0.5 \). Experimental values of the variances \( V_{z1}, V_{z2} \), found under \( \tilde{\lambda}_1 = 0.1, \tilde{\lambda}_2 = 0.9 \) and \( l_0 = 0.125, 0.25, 0.5 \), are designated by squares, crosses and rhombuses, accordingly.

While determining the experimental values of the missing probability \( \beta \) and of the variances \( V_1, V_{z1}, V_{z2} \) during simulation, it is assumed for definiteness that \( \mu = 1000 \) and \( q_0 = 1 \). However, the similar dependences are observed within a wide range of values of the parameters \( \mu \) and \( q_0 \), at least, for the case when \( \mu \geq 200 \) and \( q_0 \geq 0 \).
Finally, in Fig. 7, the theoretical dependences of the normalized conditional variance \( V_q(q_0) = V(d_m|d_0)/N_0^2 \) of the estimate \( d_m \) (29) are illustrated. The curve 1 is calculated according to the formula (42) for \( \mu = 200,\ l_0 = 0.125,\ 2 - \mu = 500,\ l_0 = 0.25,\ 3 - \mu = 1000,\ l_0 = 0.5 \). Corresponding experimental values of the variance \( V_q \) are designated by squares, crosses, rhombuses. During simulation it is assumed that \( \tilde{\Lambda}_1 = 0.1,\ \tilde{\Lambda}_2 = 0.9,\ |\Delta z| = 3 \). However, similar experimental dependences are obtained under any values of \( |\Delta z| \geq 0 \).

From the conducted analysis and Figs. 2-7 follows that the theoretical dependences for probabilities \( \alpha \) (19), \( \beta \) (25), (26) and variance \( V(l_m|l_0) \) (41) already agree quite well with the experimental data under \( \mu \geq 200,\ q_0 \geq 0,\ \tilde{\Lambda}_1 \geq 0.04,\ \tilde{\Lambda}_2 \leq 0.96 \) and \( |\Delta z| \geq 0 \). And if \( \tilde{\Lambda}_1 \geq 0.1,\ \tilde{\Lambda}_2 \leq 0.9 \), then, beginning with \( |\Delta z| \geq 10\...20 \), the simpler formula (32) can be used for calculating the variance of the estimate of the moment of stepwise change. Formulas (42) for the variances of estimates \( a_{1m},\ a_{2m} \) well approximate the experimental data under \( \mu \geq 200,\ q_0 \geq 0,\ \tilde{\Lambda}_1 \geq 0.1,\ \tilde{\Lambda}_2 \leq 0.9 \) and \( |\Delta z| > 15\...20 \), and for the variance of the estimate \( d_m \) – under \( \mu \geq 200,\ q_0 \geq 0.1 \).

5 Conclusion

In order to detect the moment of the stepwise change in the fast-fluctuating Gaussian process and to measure its jumping and constant parameters, the maximum likelihood method can be effectively applied. This approach allows us
to obtain the algorithms for determining stepwise change in the statistical characteristics of the random process in the conditions of parametrical prior uncertainty, while neglecting the values of an order of correlation time of the analyzed random process. These algorithms are technically the simplest ones in comparison with the common analogues. Using the local Markov approximation method the closed analytical expressions for the efficiency characteristics of the specified algorithms can be written down.

By the methods of statistical simulation, it is established that the obtained theoretical results agree well with the corresponding experimental data in a wide range of parameters values of the realization of the observable data. Additional researches show that the detectors and the measurers synthesized by means of the offered approach can also be used in the analysis of the stepwise changes of the statistical characteristics of the non-Gaussian random processes without large losses in performance.

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References


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