Abstract

The paper deals with a continuous stochastic model as a non-homogeneous bivariate process of birth-death, from which partial differential equations are deduced using the probability generating function (p.g.f.), the cumulant generating function (c.g.f.), and a differential equations system to the initial stages; average, variance and covariance are presented. Finally, a two-adjustments simulation system is shown.

Keywords: Continuous stochastic process, Mathematical model, Dengue dynamics, Aedes aegypti

1 Introduction

Dengue is a viral disease transmitted to humans by infected mosquitoes of the Aedes genus, being aegypti the most important one. This tropical disease is common in tropical regions. Also, it could be sporadic in subtropical and neartic zones [11, 12, 10, 14].

Aedes aegypti that is haematophagous and anthropophilic spreads out in urban and rural territories. It survives and resists to diapausa and starvation,
making it really strong and with continuous presence for long time periods. During wet season the mosquito population tends to increase. There are two well-known growth stages in the mosquitoes life cycle: the aquatic stage (includes eggs, larvae, and pupa), and mature or aerial stage \[11, 12, 10, 14\].

Mathematical models used to study Dengue dynamics, biological, and ecological phenomena are associated with their transmission and control. On the other hand, a model for stochastic process is formulated through differential equation systems that allow us to describe the growth dynamics of the \textit{A. aegypti} immature stages. Such system is deduced by applying the probability theory. In this way, Gani and Tin (1989) have extended the results of the birth-death processes in the population of both sexes to the case of including female migration \[4\]. Tapaswi and Roychoudhury (1985) have explicitly found the probability-generating function of the joint distribution, from which, it is obtained each probability distribution and the different stages \[8\]. Lounes (1989) has generalized the known results of epidemiological models in a two-type population introducing time-dependent parameters, analytic expressions are obtained to the initial stages \[9\]. Also, Ball and O’neill (1993) have described a deterministic and stochastic epidemic propagation model with incidence variations and they have studied the effect of the variability on the susceptibility \[2\].

Moreover, Lounes and Arazoza (1999) have reported a couple of models including immigration: a deterministic one and a stochastic one for a birth-death process \[3\], with a two-type population. They consider a stochastic case (applied to AIDS epidemic in Cuba) with time-dependent rates as well as the constant ones; Chen and Bokka (2005) have analyzed, modeled and simulated the spreading of that infectious disease using a non-linear stochastic model in epidemiology \[13\]; Imane, Jamal et al. (2014) have reported a stochastic and numeric analysis of a SIR model \[1\].

## 2 Stochastic model formulation

We have considered the population growth dynamics of the \textit{A. aegypti} as a non-homogeneous bivariate continuous stochastic process of birth-death \[7\]: \[((X(t),Y(t)) : t \geq 0), \] where \(X(t),Y(t)\) are random variables. However, the flows among stages along the time are Poisson-like.

The axioms that drive to the transition process between stages are: a) The process \((M(t) : t \leq 0)\) is Markovian, i.e. for any range \(t_1 < t_2 <, \ldots, < t_m\). b) The conversion number from one stage to other only depends on \((t,t + \Delta t)\), so then, it is temporarily homogeneous, where \(\Pi(\Delta t)\) and \([\Pi(t + \Delta t) - \Pi(t)]\)
are identically distributed. c) The probability of birth-death is proportional to \((t, t + \Delta t)\), if this interval is small enough and the occurrence probabilities of two or more transitions is zero during the interval of time \((t, t + \Delta t)\), then, only one transition is likely to occur in this period.

The variables and parameters of the model are: \(X(t)\): Number of adult mosquitoes at time \(t\), \(Y(t)\): Number of immature stages at time \(t\), \(\omega\): Daily developing rate of the immature stages to the mature ones, \(\epsilon\): Daily reproduction rate of the mosquitoes, \(\delta\): Mortality rate per day of the immature stages, \(\alpha\): Average reproduction rate of the mosquitoes (\textit{aedes} sp), \(\beta\): seasonal oscillation amplitude in the mosquito reproduction rate, \(\vartheta\): frequency of the mosquito proliferation cycle, \(\varphi\): Phase angle to adjust the season peak for mosquitoes [6].

### 3 Infinitesimal transition probabilities

According to the transitions of the process, the following equations are obtained:

\[
\begin{align*}
P\{X(t + \Delta t) = x + 1, Y(t + \Delta t) = y - 1/\ast\} &= \omega y \Delta t + o(\Delta t) \quad (1) \\
P\{X(t + \Delta t) = x - 1, Y(t + \Delta t) = y/\ast\} &= \epsilon x \Delta t + o(\Delta t) \quad (2) \\
P\{X(t + \Delta t) = x, Y(t + \Delta t) = y + 1/\ast\} &= \delta y \Delta t + o(\Delta t) \quad (3) \\
P\{X(t + \Delta t) = x, Y(t + \Delta t) = y- 1/\ast\} &= \delta_y \Delta t + o(\Delta t) \quad (4) \\
P\{X(t + \Delta t) = x, Y(t + \Delta t) = y/\ast\} &= 1 - \{\omega y + \epsilon x + f(t)x + \delta_y\} \Delta t + o(\Delta t) \quad (5)
\end{align*}
\]

with \(\ast = (X(t) = x, Y(t) = y)\) and initial conditions \(X(0) = x_0, Y(0) = y_0\) and \(\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0\).

By definition,

\[
p_{x,y}(t + \Delta t) = \sum_{x',y'} P\{X(t + \Delta t) = x, Y(t + \Delta t) = y/\ast\}p_{x',y'}(t) \quad (6)
\]

with \(\ast = (X(t) = x', Y(t) = y')\). Substituting the infinitesimal probability equations in (6), we have:

\[
\begin{align*}
p_{x,y}(t + \Delta t) &= \omega(y + 1)\Delta t p_{x-1,y+1}(t) + \epsilon(x + 1)\Delta t p_{x+1,y}(t) f(t) x \Delta t p_{x,y-1}(t) \\
&+ \delta(y + 1)\Delta t p_{x,y+1}(t) + \delta_y \Delta t p_{x,y}(t) - \omega y \Delta t p_{x,y}(t) - \epsilon x \Delta t p_{x,y}(t) - f(t) x \Delta t p_{x,y}(t) \quad (7)
\end{align*}
\]

Substracting \(p_{x,y}(t)\), dividing by \(\Delta t\) and taking \(\Delta t \to 0\), the differential-difference equation is obtained

\[
\frac{dp_{x,y}(t)}{dt} = \omega(y + 1)p_{x-1,y+1}(t) + \epsilon(x + 1)p_{x+1,y}(t) + f(t)xp_{x,y-1}(t) + \\
+ \delta(y + 1)p_{x,y+1}(t) - \omega yp_{x,y}(t) - \epsilon xp_{x,y}(t) - f(t)xp_{x,y}(t) - \delta yp_{x,y}(t) \quad (8)
\]
4 Generating functions (g.f.)

The probability generating function (p.g.f.) $\psi(u,v,t)$ to the bivariate process $((X(t), Y(t)): t \geq 0)$ is defined as follows:

$$\psi(u,v,t) = \sum_{x,y} u^x v^y p_{x,y}(t)$$  \hspace{1cm} (9)

from which

$$\frac{\partial \psi}{\partial t} = \sum_{x,y} u^x v^y \frac{dp_{x,y}(t)}{dt}$$  \hspace{1cm} (10)

Substituting (8) into (10),

$$\frac{\partial \psi}{\partial t} = \sum_{x,y} u^x v^y \left\{ \omega(y+1)p_{x-1,y+1}(t) + \epsilon(x+1)p_{x+1,y}(t) + f(t)xp_{x,y-1}(t) 
+ \delta(y+1)p_{x,y+1}(t) - \omega yp_{x,y}(t) - \epsilon xp_{x,y}(t) - f(t)xp_{x,y}(t) - \delta yp_{x,y}(t) \right\}$$  \hspace{1cm} (11)

from the functions (9) and (10) the equation (11) is transformed to:

$$\frac{\partial \psi}{\partial t} = (\epsilon - \epsilon u - f(t)u + f(t)uv) \frac{\partial \psi}{\partial u} + (\omega u + \delta - \omega v - \delta v) \frac{\partial \psi}{\partial v}$$  \hspace{1cm} (12)

which is the partial differential equation for the (p.g.f.), with initial condition $\psi(u,v,0) = u^{x_0}$.

In a bivariate process, the cumulant-generating function (c.g.f.) is defined as the natural algorithm of the p.g.f., $\psi(u,v,t)$ with $u = e^\theta$ and $v = e^\gamma$: $K(\theta, \gamma, t) = \ln \psi(u,v,t)$. Applying partial derivation with respect to $\theta$ and $\gamma$, and replacing them in (12), we obtain the PDE corresponding to the c.g.f.

$$\frac{\partial K}{\partial t} = (\epsilon e^{-\theta} - \epsilon - f(t) + f(t)e^\gamma) \frac{\partial K}{\partial \theta} + (\omega e^{\theta-\gamma} + \delta e^{-\gamma} - (\delta + \omega)) \frac{\partial K}{\partial \gamma}$$  \hspace{1cm} (13)

with initial conditions $K(\theta, \gamma, 0) = x_0 \theta$.

5 Differential Equations to the three first-moments

By definition,

$$K(\theta, \gamma, t) = \sum_{i,j \geq 0} \kappa_{ij}(t) \frac{\partial^i \gamma^j}{i! j!} = \theta k_{10}(t) + \frac{\theta^2}{2!} k_{20}(t) + \gamma k_{01}(t) + \frac{\gamma^2}{2!} k_{02}(t) + \theta \gamma k_{11} + \ldots$$  \hspace{1cm} (14)

where, $k_{10}(t) = \mu_X(t)$, $k_{01}(t) = \mu_Y(t)$, $k_{20}(t) = \sigma_X^2(t)$, $k_{02}(t) = \sigma_Y^2(t)$, $k_{11}(t) = \text{cov}_{XY}(t)$. 
Stochastic modelling of *Aedes aegypti*

Upon applying partial derivation to (14) with respect to \( t \), \( \theta \), \( \gamma \), and also, the series \( e^{\theta} \), \( e^{-\theta} \), \( e^{\gamma} \) and \( e^{-\gamma} \). Substituting these series into (13) we have

\[
\theta^2 \frac{d\mu}{dt} + \frac{\theta^2}{2} \frac{d\sigma^2}{dt} + \gamma \frac{d\mu}{dt} + \frac{\gamma^2}{2} \frac{d\sigma^2}{dt} + \theta \gamma \frac{dcov_{XY}}{dt} + \ldots
\]

\[
= \left( -\epsilon \theta + \epsilon \frac{\theta^2}{2} + \gamma f(t) + \frac{\gamma^2}{2} f(t) \right) \left( \mu_X + \theta \sigma^2_X + \gamma cov_{XY} + \ldots \right)
\]

\[
+ \left( \left( \omega + \omega \theta + \omega \frac{\theta^2}{2} + \ldots \right) \left( 1 - \gamma + \frac{\gamma^2}{2} - \ldots \right) - \delta \gamma + \delta \frac{\gamma^2}{2} - \omega \right)
\]

\[
(\mu_Y + \gamma \sigma^2_Y + \theta cov_{XY} + \ldots)
\]

Finally, upon equaling the coefficients of \( \theta \), \( \theta^2 \), \( \gamma \) and \( \gamma^2 \) the differential equations to the three first-moments (average, variance, and covariance) are obtained:

\[
\frac{d\mu}{dt} = \omega \mu_Y - \epsilon \mu_X \quad (16)
\]

\[
\frac{d\mu_Y}{dt} = f(t)\mu_X - (\delta + \omega)\mu_Y \quad (17)
\]

\[
\frac{d\sigma^2}{dt} = \epsilon \mu_X + \omega \mu_Y + 2\omega cov_{XY} - 2\epsilon \sigma^2_X \quad (18)
\]

\[
\frac{d\sigma^2_Y}{dt} = f(t)\mu_X + (\delta + \omega)\mu_Y + 2f(t)cov_{XY} - 2(\delta + \omega)\sigma^2_Y \quad (19)
\]

\[
\frac{dcov_{XY}}{dt} = \omega \sigma^2_Y + f(t)\sigma^2_X - \omega \mu_Y - \epsilon cov_{XY} - (\delta + \omega) cov_{XY} \quad (20)
\]

where, \( f(t) = \alpha [1 - \beta \sin(\theta t + \varphi)] \) with the initial conditions \( \mu_x(0) = \mu_{x_0}, \mu_y(0) = \mu_{y_0}, \sigma^2_x(0) = 0, \sigma^2_y(0) = 0 \) and \( cov_{xy}(0) = 0 \).

### 6 Results and conclusion

Simulation of the system has been realized using the MatLab software, the values of the parameters and the periodic growth function \( f(t) \) were considered as the ones proposed by Caetano and Yoneyama [6].

We have seen a non-monotonic behavior in both populations due to the periodic function \( xx \), with a rapid decrease in the immature-stages number, which is due to the high developing rate. The left side of Fig 1 shows that both populations tend to stabilize in a value under 50 mosquitoes in approximately 100 days. This result is not surprising since the initial conditions are 100 adult and 500 immature mosquitoes.
Figure 1: Average population, variance, covariance and mosquitoes variation coefficient, with $\omega =0.1[15]$, $\epsilon =0.1[5]$, $\delta = 0.096[6]$, $\alpha =0.4$, $\beta =0.4[6]$, $\vartheta = 2\pi/52[6]$, $\varphi =0[6]$, $X(t) = (\mu_X, \mu_Y, \sigma^2_X, \sigma^2_Y, \text{Cov}_{XY})$.

The right side of Fig 1 shows the variance of both populations, as well as the covariance between each other, from which, we conclude that both populations have a high correlation.

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