Theoretical Analysis of an Optimal Control Model

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Abstract

We propose and analyze a mathematical model for the mechanical control of the dengue transmitter Aedes aegypti. We are interested in the diminish of the charge capacity of the breeding places ($I$) with the minimum economic effort, as direct consequence a decreasing of the mature mosquitoes ($m$) and immature stages ($a$) population. To achieve this goal we have used an optimal control problem, with an analysis realized using the Pontryagin’s Maximum Principle.

Keywords: Dengue, Aedes aegypti, Mechanical control, Optimal control, Pontryagin’s Maximum Principle

1 Introduction

Dengue is a viral disease transmitted to humans by Aedes aegypti mosquitoes. It is endemic in tropical regions and sporadic in the subtropical ones. This viral disease was recognized clinically in America, Asia, and Africa since the
XVIII century [4, 11, 14]. The very first dengue transmission by the A. *aegypti* mosquito was proved in 1918. After that, in 1944 it was achieved the dengue causal-agent isolation [11, 14].

Chemical (larvicides and adulticides), mechanical, biological, and genetic methods are used to control the A. *aegypti* mosquitoes. The mechanical control method consists of all the physical activities oriented to disturb the biological aquatic stage of the mosquitoes. Elimination of the breeding places affects directly to the population growth of the mosquitoes and of course the dengue incidence [4, 14].

We have constructed and analyzed mathematical models based on ordinary differential equation systems, partial differential equations, integro-differential equations and topological lattices to study: Transmission dynamics of the classic dengue with a constant and variable human population [12, 6, 3]; Space-time transmission dynamics [7]; The stochastic population dynamics [10]; The mosquito control including: the chemical control effects in the constant mortality rates [5, 2]; and applying the Pontryagin’s Maximum Principle [8, 9, 1].

2 The optimal control model

The model is proposed and analyzed for the mechanical control of the mosquitoes, with the following variables and parameters: $m$ is the total average number of mature mosquitoes at time $t$, $a$ is the average number of immature stages (eggs, larvae, pupas) in the aquatic phase at time $t$, $\phi$ the number of mosquito eggs per day, $I$ the charge capacity of the breeding places at time $t$, $\epsilon$ the natural mortality rate of the mature mosquitoes, $\pi$ represents the natural mortality rate of the immature stages, $\omega$ is the developing rate from the immature stages to the mature ones, and $u$ is the breeding place elimination cost.

We intend to minimize the quadratic functional

$$ J(u_1) = \int_0^{t_1} \left\{ \eta_1 m^2(t) + \eta_2 a^2(t) + \eta_3 u_1^2(t) \right\} dt \longrightarrow \min_{u_1 \in \Gamma} \quad (1) $$

with $\eta_i > 0$, $i = 1, 2, 3$, $u_1 \in \Gamma \subset L^2([0,t_1])$ and the stage condition $I > 0$, for a fixed time $t_1$, and connected to the system,

$$ \frac{dm}{dt} = \omega a - \epsilon m \quad (2) $$
$$ \frac{da}{dt} = \phi m \psi_3 \left( \frac{a}{I} \right) - (\pi + \omega) a \quad (3) $$
$$ \frac{dI}{dt} = -u_1 \quad (4) $$
Theoretical analysis of an optimal control model

where \( \psi_3(-\infty, +\infty) \subset (0,1) \), \( \lim_{x \to -\infty} \psi_3(x) = 1 \), \( \lim_{x \to +\infty} \psi_3(x) = 0 \), \( \psi_3 \) decreasing and with initial conditions \( m(0) = m_0 \), \( a(0) = a_0 \) and \( I(0) = I_0 \).

Equations (2) and (3) describe the dynamics of the mosquito population and the last equation (4), the velocity equation related to the charge capacity of the breeding places in terms of the control, \( u_1 \): represents the necessary economic inversion to diminish the breeding place capacity. We analyze the system (2-4) in the neighborhood of the stable stationary solution \((\hat{m}, \hat{a}, \hat{I})\) considering the appropriate deviations to the stationary solution

\[
\Delta m = m - \hat{m} \quad , \quad \Delta a = a - \hat{a} \quad , \quad \Delta I = I - \hat{I}
\]

and the deviation of the control \( \Delta u = u - \hat{u} \), characterizing the system in deviations with the form

\[
\frac{dx}{dt} = Ax + B\Delta u
\] (5)

where

\[
A = \frac{\partial f(\hat{m}, \hat{a}, \hat{I}, u^0)}{\partial x} \quad , \quad B = \frac{\partial f(\hat{m}, \hat{a}, \hat{I}, u^0)}{\partial u}
\]

Afterwards, we have the matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 0
\end{bmatrix}
\] (6)

where

\[
a_{11} = \frac{\partial f_1(\hat{m}, \hat{a}, \hat{I}, u^0)}{\partial m} = -\epsilon \quad , \quad a_{12} = \frac{\partial f_1(\hat{m}, \hat{a}, \hat{I}, u^0)}{\partial a} = \omega
\]

\[
a_{21} = \frac{\partial f_2(\hat{m}, \hat{a}, \hat{I}, u^0)}{\partial m} = \phi \psi_3\left(\frac{\hat{a}}{\hat{I}}\right) \quad , \quad a_{22} = \frac{\partial f_2(\hat{m}, \hat{a}, \hat{I}, u^0)}{\partial a} = \phi \frac{\hat{m} \partial \psi_3(\hat{a}/\hat{I})}{\hat{a}} - (\pi + \omega)
\]

and

\[
B = \begin{bmatrix}
0 \\
0 \\
b_3
\end{bmatrix}
\]

where \( b_3 = \frac{\partial f(\hat{m}, \hat{a}, \hat{I}, u^0)}{\partial u_1} = -1 \). Therefore, the system in deviations is

\[
\dot{\Delta} m = -\epsilon \Delta m + \omega \Delta a
\] (7)

\[
\dot{\Delta} a = \phi \psi_3\left(\frac{\hat{a}}{\hat{I}}\right) \Delta m + \left(\phi \frac{\hat{m} \partial \psi_3(\hat{a}/\hat{I})}{\hat{a}} - \pi - \omega\right) \Delta a + \phi \frac{\hat{m} \hat{a} \partial \psi_3(\hat{a}/\hat{I})}{\hat{I}^2} \Delta I
\] (8)
\[ \dot{\Delta I} = -\Delta u \]  

(9)

The system (5) is completely controllable at a time \( t_1 > 0 \), if the established initial and final conditions \( x^0, x^1 \in \mathbb{R}^n \) there exist an \( u \in L^2(0, t_1; \mathbb{R}^m) \), so that, the solution of (5) satisfy \( x(t_1) = x^1 \). Namely, the goal is to lead the solution from an initial state \( x^0 \) to a final one \( x^1 \) at a time \( t_1 \) for the system (5) through the control \( u \). The following classic result, because R. E. Kalman, gives solution to the problem of full controllability of a finite-dimension linear system as the system (5).

System (5) is completely controllable at a time \( t_1 \), if and only if

\[ \text{rang}[B, AB, ..., A^{n-1}B] = n \]  

(10)

namely, \( \text{det}[B, AB, ..., A^{n-1}B] \neq 0 \).

Here, \([B, AB, ..., A^{n-1}B]\) is called the controllability matrix. For the system (2), (3), and (4), \( n = 3 \) and \( m = 1 \), the controllability matrix is

\[ C = [B, AB, A^2B] \]

that is to say,

\[ C = \begin{pmatrix} 0 & 0 & a_{12}a_{23}b_3 \\ 0 & a_{23}b_3 & a_{22}a_{23}b_3 \\ b_3 & 0 & 0 \end{pmatrix} \]

and the system is completely controllable, if and only if, the range of the controllability matrix is full, namely, if \( \text{rang}[B, AB, A^2B] = 3 \). After,

\[ \text{det} \left( \begin{pmatrix} 0 & a_{12}a_{23}b_3 \\ a_{23}b_3 & a_{22}a_{23}b_3 \\ b_3 & 0 & 0 \end{pmatrix} \right) = -a_{12}a_{23}^2b_3^2 = \omega \left( \frac{\phi \partial \psi_3(z)}{T^2} \right)^2 > 0 \]

So that, the system is fully controllable.

### 2.1 Pontryagin’s Maximum Principle

In 1956 L. S. Pontryagin’s stated a criterion to find optimal controls, that is called maximum principle. We write the adjoint system and the Pontryagin’s function for this system. The conjugate system is given by

\[ \frac{d\rho}{dt} = -\left( \frac{\partial f}{\partial y} \right)^T \rho \]  

(11)

where, \( y = (J_0, m, a, I)^T \), \( \rho = (\rho_1, \rho_2, \rho_3, \rho_4)^T \), \( f = (f_1, f_2, f_3, f_4)^T \)
Afterwards,

$$-\left( \frac{\partial f}{\partial y} \right)^T = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-2\eta_1 m & \epsilon & -\phi \psi_3 \left( \frac{a}{I} \right) & 0 \\
-2\eta_2 a & -\omega & -\phi m \frac{\partial \psi_3 \left( \frac{a}{I} \right)}{\partial a} + \pi + \omega & 0 \\
0 & 0 & \phi m a \frac{\partial \psi_3 \left( \frac{a}{I} \right)}{\partial T} & 0
\end{pmatrix}$$

(12)

so then, the adjoint system of (1) to (4) according with (11) is:

$$\frac{d\rho}{dt} = \begin{pmatrix}
\frac{d\rho_1}{dt} \\
\frac{d\rho_2}{dt} \\
\frac{d\rho_3}{dt} \\
\frac{d\rho_4}{dt}
\end{pmatrix} = \begin{pmatrix}
0 \\
-2\eta_1 \rho_1 m + \epsilon \rho_2 - \phi \psi_3 \left( \frac{a}{I} \right) \rho_3 \\
-2\eta_2 \rho_1 a - \omega \rho_2 + \left( -\phi m \frac{\partial \psi_3 \left( \frac{a}{I} \right)}{\partial a} + \pi + \omega \right) \rho_3 \\
-\phi m a \frac{\partial \psi_3 \left( \frac{a}{I} \right)}{\partial T} \rho_3
\end{pmatrix}$$

(13)

namely

$$\frac{d\rho_2}{dt} = -2\eta_1 \rho_1 m + \epsilon \rho_2 - \phi \psi_3 \left( \frac{a}{I} \right) \rho_3$$

(14)

$$\frac{d\rho_3}{dt} = -2\eta_2 \rho_1 a - \omega \rho_2 + \left( -\phi m \frac{\partial \psi_3 \left( \frac{a}{I} \right)}{\partial a} + \pi + \omega \right) \rho_3$$

(15)

$$\frac{d\rho_4}{dt} = -\phi m a \frac{\partial \psi_3 \left( \frac{a}{I} \right)}{\partial T} \rho_3$$

(16)

The Pontryagin’s function for the adjoint variables $\rho_i, i = 1, ..., 4$ of the control problem is:

$$H = \rho^T \cdot f$$

$$H = \rho_1 (\eta_1 m^2 + \eta_2 a^2 + \eta_3 u_1^2) + \rho_2 (\omega a - \epsilon m) + \rho_3 \{ \phi m \psi_3 \left( \frac{a}{I} \right) - (\pi + \omega) a \} - \rho_4 u_1$$

(17)

From the first order control condition $\frac{\partial H}{\partial u} = 0$, we obtain the optimal control,

$$\frac{dH}{du_1} = 2\rho_1 \eta_3 u_1 - \rho_4$$

(18)

Therefore

$$u_1 = \frac{\rho_4}{2\rho_1 \eta_3} = -\frac{1}{2\eta_3} \rho_4, \quad \eta_3 > 0.$$

(19)
Then, we substitute the control $u_1$ in the system of equations (2)-(4) and grouping together with (14), (15) and (16) for the following boundary problem.

\[
\frac{dm}{dt} = \omega a - \epsilon m \quad (20)
\]
\[
\frac{da}{dt} = \phi m \psi_3 \left( \frac{a}{I} \right) - (\pi + \omega) a \quad (21)
\]
\[
\frac{dI}{dt} = \frac{1}{2\eta_3} \rho_4 \quad (22)
\]
\[
\frac{d\rho_2}{dt} = 2\eta_1 m + \epsilon \rho_2 - \rho_3 \phi \psi_3 \left( \frac{a}{I} \right) \quad (23)
\]
\[
\frac{d\rho_3}{dt} = 2\eta_2 a - \omega \rho_2 - \phi \frac{m}{I} \rho_3 \frac{\partial \psi_3 \left( \frac{q}{I} \right)}{\partial a} + (\pi + \omega) \rho_3 \quad (24)
\]
\[
\frac{d\rho_4}{dt} = -\rho_3 \phi \frac{ma \partial \psi_3 \left( \frac{q}{I} \right)}{I^2} \frac{\partial I}{\partial \rho_4} \quad (25)
\]

with initial and final conditions $m(0) = m_0$, $a(0) = a_0$, $I(0) = I_0$ and $\rho_2(t_1) = 0$, $\rho_3(t_1) = 0$, $\rho_4(t_1) = 0$.

### 3 Conclusions

The discussed optimal control model generates a family of models according to the mathematical shape of the $\psi_3$. A trivial form of that function is $\psi_3 \left( \frac{a}{I} \right) = 1 - \frac{a}{I}$; so then, the optimal control model results in

\[
\frac{dm}{dt} = \omega a - \epsilon m \quad (26)
\]
\[
\frac{da}{dt} = \phi m (1 - \frac{a}{I}) - (\pi + \omega) a \quad (27)
\]
\[
\frac{dI}{dt} = -u_1 \quad (28)
\]

this is a dynamic system integrated to the functional as (1) with initial conditions $m(0) = m_0 \geq 0$, $a(0) = a_0 \geq 0$ and parameters $\omega, \epsilon, \pi, I, \phi \geq 0$.

We have supposed a set of initial conditions belonging to the biological sense region $\Sigma_B$. Namely,

\[
\Sigma_B = \{(m, a) : m \geq 0, 0 \leq a \leq I \}
\]

the coexistence stationary solution with mosquitoes and immature stages is:

\[
(\hat{m}, \hat{a}) = \left( 1 - \frac{\epsilon(\pi + \omega)}{\phi \omega} \right) I \left( \frac{\omega}{\epsilon}, 1 \right)
\]
Talking about the mosquito growth threshold \( R_M = \frac{\phi \omega}{\epsilon (\pi + \omega)} \), the coexistence stationary solution has the following form.

\[
(\hat{m}, \hat{a}) = \frac{I}{R_M} (R_M - 1) \left( \frac{\omega}{\epsilon}, 1 \right)
\]

only with biological sense where \( R_M \geq 1 \). It is local and asymptotically stable if \( R_M > 1 \).

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