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# Bifurcations in the Dynamics of Rosenzweig-MacArthur Predator-Prey Model Considering Saturated Refuge for the Preys

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#### Abstract

In this work, we use an analytical approach to study the dynamic consequences of refuge use by the prey in the Rosenzweig-MacArthur predator-prey model with the refuge function proposed by Almanza-Vasquez. We will evaluate the effects with regard to the local stability of equilibrium points in the first quadrant. We show that there is a trend from limit cycles through non-zero stable points.

#### Mathematics Subject Classification: 34A34

**Keywords:** Stability, Limit cycles, Bifurcation, Predator–prey models, Refuge

## 1 Introduction

In mathematical ecology, one of the popular models is a model consisting of two difference species where one of them provides food to the other. Many preys respond to the attacks of the predators looking for such space refuges. The effect itself of refuge use on the population growth is complex in nature, but for modeling purposes it can be understood as the reduction of prey mortality due to reduction in predation success. The refuges affect positively the population growth of preys and negatively that of predators. A more relevant behavioral trait that affects the dynamics of predator-prey systems is the use of spatial refuges by the prey. Spatial refuges are found where environmental heterogeneity provides less-accessible sites for predators in which a number of preys can stay, at least temporarily. In this way, some fraction of the prey population is partially protected against predators and we assume that the refuge is a physical location in which preveither live or temporally hide [2]. The knowledge of the impact of prey refuge use by a fraction of prey-population is relevant in the context of bioeconomic and conservation management, because it helps in regulating the harvesting activity in the ecosystem and management of reserves or non-take zones; also it is essential for conservation of endangered species creating protected areas (reserves) for preserve them [3]. In earlier works it has been claimed that the prey refuge use exerts a stabilizing effect in the dynamics of the interacting populations. We show what physical capacity of refuge it influences in the existence and stability the unique equilibrium point at interior of the first quadrant. However, González-Olivares and Ramos-Jiliberto, discard the common conclusion that the use of the shelter by the population of prev always leads to stability as considering the same assumptions in the model Rosenzweig-MacArthur obtained which trajectories can oscillate for some parameter values [4]. We denote by X(t) = X and Y(t) = Y the population sizes of preys and predators, respectively for  $t \geq 0$ , considered as continuous variables that can represent density, biomass or quantity of each population's individuals. The Rosenzweig-MacArthur predator prey model is

$$\chi_{\varphi} = \begin{cases} \frac{dX}{dt} = r(1 - \frac{X}{K})X - \frac{qXY}{X+a}\\ \frac{dY}{dt} = b(\frac{pX}{X+a} - c)Y \end{cases}$$
(1)

 $\varphi = (a, b, c, K, p, q, r) \in \mathbb{R}^7_+$  is usually a vector of biological parameters. The parameters have the following biological meanings:

 $\boldsymbol{r}$  is the intrinsic per capita prey growth rate;

K is the prey environmental carrying capacity;

q is the maximal per capita predator consumption rate;

a is the amount of prey needed to achieve one-half of q;

b is the efficiency with which predators convert consumed prey into new predators;

c is the natural per capita death predator rate.

If  $X_r(t) = X_r$ , a quantity of prey population that occupies a refuge (heterogeneity of the means), the quantity of preys in refuge, then the quantity of preys that interact with the predators is  $X - X_r$ . The model (1) is transformed

$$X_{\mu}^{X_r} = \left\{ \begin{array}{l} \frac{dX}{dt} = r\left(1 - \frac{X}{K}\right) X - \frac{q(X - X_r)Y}{(X - X_r) + a} \\ \frac{dY}{dt} = b\left(\frac{p(X - X_r)}{(X - X_r) + a} - c\right) Y \end{array} \right\}$$
(2)

Almanza-Vasquez [1] analyzed the population consequences of refuge use in the Lotka-Volterra model with self-limitation, assuming that the amount of prey in refuge using a saturated function that it is growing monotonously in the way

$$X_r = \frac{\alpha X}{X + \beta}$$

Her  $\alpha$  represents the maximum physical capacity of refuge and where the population's fraction in cover is falling in the way.  $\beta$  is the quantity of necessary preys to reach half of the maximum capacity  $\alpha$ . Considering  $X_r = \frac{\alpha X}{X+\beta}$ , the model  $X_{\mu}^{X_r}$  is transformed

$$X_{\mu}^{\alpha,\beta} = \begin{cases} \frac{dX}{dt} = r\left(1 - \frac{X}{K}\right) X - \frac{q\left(X - \frac{\alpha X}{X + \beta}\right) Y}{\left(X - \frac{\alpha X}{X + \beta}\right) Y + a} \\ \frac{dY}{dt} = b\left(\frac{p\left(X - \frac{\alpha X}{X + \beta}\right)}{\left(X - \frac{\alpha X}{X + \beta}\right) + a} - c\right) Y \end{cases}$$
(3)

 $\mu = (\alpha, \beta, r, K, q, p, a, b, c) \in \mathbb{R}^9_+$  is a vector of biological parameters.

## 2 The model

Considering the function proposed in the model obtains the field vectorial  $X^{\alpha,\beta}_{\mu}$  described by the autonomous system of differential equations (3) and following the methodology used in [8], we make a reparametrization of the vector field  $X^{\alpha,\beta}_{\mu}$  or the system (3) including changes of variables and a time rescaling given by the diffeomorphism

$$\varphi: (R^+)^2 \times R \to (R^+)^2 \times R$$
, such that  
 $(N, P, \tau) \to (X, Y, t)$  where

$$\begin{split} \varphi(N,P,\tau) &= (KN, \frac{rK}{q}P, \frac{((N-(A-B))N+\frac{E}{D}(N+B))\tau}{r}) = (X,Y,t), \ A = \frac{\alpha}{K}, \ B = \frac{\beta}{K}, \\ C &= \frac{bp}{r}, \ D = \frac{bc}{r}, \ E = \frac{bc}{r}(\frac{a}{K}). \ \text{And} \ \frac{dN}{d\tau} = \frac{1}{K}(\frac{dX}{dt}) \to 0. \\ \text{The vector field in the new coordinates is } X_{\eta} = \varphi \circ X_{\mu}^{\alpha,\beta}, \ \text{and associated second-order differential equations system is the following Kolmogorov-type system:} \\ \text{If } \overline{C} = \frac{E}{D}, \ H = BC - AC + AD - BD - \overline{C}, \ X_{\eta}^{A} \ \text{is defined by the system} \end{split}$$

$$\frac{dN}{d\tau} = N \left[ -N^3 + (1+\alpha)N^2 + (-\alpha - \overline{C}B)N + \overline{C}B - (N-A+B)P \right], \quad (4)$$
$$\frac{dP}{d\tau} = P \left[ (C-D)N^2 + HN - \overline{C}B \right],$$

with  $\alpha = B - A + \overline{C}$ . Which has only five parameters, i.e.,  $\eta = (A, B, C, D, E) \in \mathbb{R}^5_+$  and the system (4) is topologically equivalent to system (3). For  $N \neq A - B$ ,  $(A - B - \overline{C})^2 - 4B\overline{C} \geq 0$ ,  $H = BC - AC + AD - BD - \overline{C}$ . Then the N-nulleclines associated to the system (4) are given by:

$$N=0,$$

$$P = \frac{(1-N)(N - (\frac{1}{2}(\alpha + \sqrt{\alpha^2 - 4B\overline{C}}))(N - (\frac{1}{2}(\alpha - \sqrt{\alpha^2 - 4B\overline{C}}))}{N + B - A}$$

where  $\alpha = (A - B - \overline{C})$ . The P-nulleclines associated to the system (4) are given by:

$$P = 0$$
$$N = \frac{-H \pm \sqrt[2]{H^2 + 4B\overline{C}(C-D)}}{2(C-D)}$$

For  $S = A - B - \overline{C} + \overline{C}B$  and  $M = 1 + A - B - \overline{C}$ . The Jacobian matrix of system (4) is

$$JX_{\eta}^{A}(N,P) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
  
$$a_{11} = -4N^{3} + 3MN^{2} - 2(S+2P)N + B\overline{C} + (A-B)P,$$
  
$$a_{12} = -N(N+B-A),$$
  
$$a_{21} = (2N(C-D) + H)P,$$
  
$$a_{22} = (C-D)N^{2} + HN - B\overline{C}.$$

Considering the isolated means, where there is not reintroduction of preys X such that overcome the prey environmental carrying capacity K.

### 3 Main Results

For system (4) we have that

**Lemma 3.1.** The set  $\Omega = \{(N, P) \in (R_0^+)^2 | 0 \le N \le 1, P \ge 0\}$  is an invariant region of vector fields.

- *Proof.* 1. As system (4) is of Kolmogorov type, then the N-axis and P-axis are invariant sets.
  - 2. If N = 1, we have that  $\frac{dN}{d\tau} = -(1 + B A)P < 0$ , and the trajectories point into region  $\Omega$ .

#### Lemma 3.2. The trajectories are bounded.

Proof. Using the Poincaré compactification.

**Lemma 3.3.** 1. If  $A < \frac{(1+B)(C-D-\overline{C})}{C-D}$ , the system (4) has tree equilibrium points:

- (a)  $(N_1^*, P_1^*) = (0, 0),$ (b)  $(N_2^*, P_2^*) = (1, 0),$ (c)  $(N_3^*, P_3^*) = \left(\frac{-H + \sqrt[2]{H^2 + 4B\overline{C}(C-D)}}{2(C-D)}, \frac{-N^{*3} + MN^{*2} - SN^* + B\overline{C}}{N^* - A + B}\right).$
- 2. If  $A \ge \frac{(1+B)(C-D-\overline{C})}{C-D}$ , the system (4) has five equilibrium points:
  - $(a) \quad (N_1^*, P_1^*) = (0, 0),$   $(b) \quad (N_2^*, P_2^*) = (1, 0),$   $(c) \quad (N_3^*, P_3^*) = \left(\frac{-H + \sqrt[2]{H^2 + 4B\overline{C}(C-D)}}{2(C-D)}, \frac{-N^{*3} + MN^{*2} - SN^* + B\overline{C}}{N^* - A + B}\right),$   $(d) \quad (N_4^*, P_4^*) = \left(\frac{1}{2}((A - B - \overline{C}) - \sqrt{(A - B - \overline{C})^2 - 4B\overline{C}}, 0), \right),$   $(e) \quad (N_5^*, P_5^*) = \left(\frac{1}{2}((A - B - \overline{C}) + \sqrt{(A - B - \overline{C})^2 - 4B\overline{C}}, 0)\right) \text{ where }$ H = D + (A - B)C.

*Proof.* Considering the equations that define system (4).

**Lemma 3.4.** 1. The singularity  $(N_1^*, P_1^*) = (0, 0)$  is saddle point for all parameter values.

2. The singularity  $(N_2^*, P_2^*) = (1, 0)$  is: Globally asymptotically stable, if  $A \ge \frac{(1+B)(C-D-\overline{C})}{C-D}$ . Saddle point, if  $A < \frac{(1+B)(C-D-\overline{C})}{C-D}$ .

*Proof.* Evaluating the Jacobian matrix we have:

1. 
$$JX_{\eta}^{A}(0,0) = \begin{pmatrix} B\overline{C} & 0\\ 0 & -B\overline{C} \end{pmatrix}$$

The eigenvalues:  $\lambda_1 = B\overline{C} > 0, \ \lambda_2 = -B\overline{C} < 0.$ 

2. 
$$JX_{\eta}^{A}(1,0) = \begin{bmatrix} A - (1+B+\overline{C}+B\overline{C}) & A - (1+B) \\ 0 & C - D + H - B\overline{C} \end{bmatrix}$$

The eigenvalues:  $\lambda_1 = A - (1 + B + \overline{C} + B\overline{C}) < 0$ ,  $\lambda_2 = C - D + H - B\overline{C}$ . We observe that the sign of  $\lambda_2$  depends on the value of the parameters like it indicates the hypothesis: If  $A > \frac{(1+B)(C-D-\overline{C})}{C-D}$ ,  $\lambda_2 < 0$  and applying the Poincaré Bendixson theorem (1,0) is globally asymptotically stable. If  $A < \frac{(1+B)(C-D-\overline{C})}{C-D}$ ,  $\lambda_2 > 0$ , (1,0) is saddle point.

For the proofs of the following theorems are considered: For every positive integer m and n have to be  $\sqrt{m^2 + n^2} \le m + n$  and relationships between the parameters:

$$\begin{split} A > B, C > D, H < 0, \\ \frac{1}{2}((A - B - \overline{C}) \pm \sqrt{(A - B - \overline{C})^2 - 4B\overline{C}}) &\leq \frac{-H + \sqrt{H^2 - 4(C - D)B\overline{C}}}{2(C - D)} \leq \frac{S + \sqrt{S^2 - 3MB\overline{C}}}{M} \\ H^2 - 4(C - D)B\overline{C} \geq 0, \\ S^2 - 3MB\overline{C} \geq 0. \end{split}$$

Furthermore  $\overline{C} \to 0$ .

**Lemma 3.5.** If  $A \ge \frac{(1+B)(C-D-\overline{C})}{C-D}$ : The singularity  $(N_4^*, P_4^*) = (\frac{1}{2}((A-B-\overline{C}) - \sqrt{(A-B-\overline{C})^2 - 4B\overline{C}}, 0)$  is globally asymptotically stable and the singularity  $(N_5^*, P_5^*) = (\frac{1}{2}((A-B-\overline{C}) + \sqrt{(A-B-\overline{C})^2 - 4B\overline{C}}, 0)$  is saddle point.

Proof. Evaluating the Jacobian matrix in  

$$(N^*, P^*) = \left(\frac{1}{2}((A - B - \overline{C}) \pm \sqrt{(A - B - \overline{C})^2 - 4B\overline{C}}, 0) \text{ we have:} \\ \begin{bmatrix} -MN^{*2} + 2SN^* - 3B\overline{C} & -N^*(N^* + B - A) \\ 0 & (C - D)N^{*2} + HN^* - B\overline{C} \end{bmatrix}$$

The eigenvalues:

$$\begin{split} \lambda_1 &= -MN^{*2} + 2SN^* - 3B\overline{C} < 0, \\ \text{for } N^* &= \frac{1}{2}((A - B - \overline{C}) - \sqrt{(A - B - \overline{C})^2 - 4B\overline{C}}, \ \lambda_1 &= -MN^{*2} + 2SN^* - 3B\overline{C} > 0, \text{ for } N^* &= \frac{1}{2}((A - B - \overline{C}) + \sqrt{(A - B - \overline{C})^2 - 4B\overline{C}}, \text{ And } \lambda_2 &= (C - D)N^{*2} + HN^* - B\overline{C} = (N^* - \frac{-H + \sqrt{H^2 + 4B\overline{C}(C - D)}}{2(C - D)})(N^* - \frac{-H - \sqrt{H^2 + 4B\overline{C}(C - D)}}{2(C - D)}) < 0 \\ \text{insomuch as } N^* < \frac{-H + \sqrt{H^2 + 4B\overline{C}(C - D)}}{2(C - D)}, \text{ this proves the lemma.} \end{split}$$

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**Theorem 3.6.** For system (4) we have the following results. The singularity  $(N_3^*, P_3^*) = \left(\frac{-H + \sqrt[2]{H^2 + 4B\overline{C}(C-D)}}{2(C-D)}, \frac{-N^{*3} + MN^{*2} - SN^* + B\overline{C})}{N^* - A + B}\right)$  is: Center, if  $A = \frac{(1+B)(C-D-\overline{C})}{C-D}$ . Spiral unstable, if  $A > \frac{(1+B)(C-D-\overline{C})}{C-D}$ . Spiral stable, if  $A < \frac{(1+B)(C-D-\overline{C})}{C-D}$ .

*Proof.* As  $-N^3 + MN^2 - SN + \overline{C}B - (N - A + B)P = 0$ . Evaluating the Jacobian matrix in  $(N_3^*, P_3^*) = \left(\frac{-H + \sqrt[2]{H^2 + 4B\overline{C}(C-D)}}{2(C-D)}, \frac{-N^{*3} + MN^{*2} - SN^* + B\overline{C}}{N^* - A + B}\right)$  we have:  $JX_n^A(N^*, P^*)$ 

$$= \begin{bmatrix} -MN^{*2} + SN^* - 3(A - B + B\overline{C})P^* & -N^*(N^* + B - A) \\ 2(C - D)N^*P^* + HP^* & 0 \end{bmatrix}$$

with  $TraceJX_{\eta}(N^*, P^*) = -MN^{*2} + SN^* - 3(A - B + B\overline{C})P^*$  and  $DetJX_{\eta}(N^*, P^*) = N^*P^*(N^* + B - A)(2N^*(C - D) + H) > 0$ , because if the above is not true,  $2N^*(C - D) + H < 0$  and so  $\frac{-H + \sqrt[2]{H^2 + 4B\overline{C}E(C - D)}}{2(C - D)} < \frac{-H}{2(C - D)}$ . And the behavior of singularity depends on the trace  $TraceJX_{\eta}^A(N^*, P^*)$ : For  $A > \frac{(1+B)(C - D - \overline{C})}{C - D}$ , it has  $\frac{S}{M} > N^* > \frac{S + \sqrt{S^2 - 12M(A - B + B\overline{C})P^*}}{2M}$  therefore  $TraceJX_{\eta}(N^*, P^*) = -MN^{*2} + SN^* - 3(A - B + B\overline{C})P^* > 0$ , and  $(N^*, P^*)$ is spiral unstable. For  $A < \frac{(1+B)(C - D - \overline{C})}{C - D}$ , it has  $\frac{S - \sqrt{S^2 - 12M(A - B + B\overline{C})P^*}}{2M} < \frac{S}{M} < N^* < \frac{S + \sqrt{S^2 - 12M(A - B + B\overline{C})P^*}}{2M}$ . Therefore  $TraceJX_{\eta}(N^*, P^*) = -MN^{*2} + SN^* - 3(A - B + B\overline{C})P^* > 0$ , and  $(N^*, P^*)$   $\frac{S}{M} < N^* < \frac{S + \sqrt{S^2 - 12M(A - B + B\overline{C})P^*}}{2M}$ . Therefore  $TraceJX_{\eta}(N^*, P^*) = -MN^{*2} + SN^* - 3(A - B + B\overline{C})P^* < 0$  and the singularity  $(N^*, P^*)$  is spiral attractor. And the behavior of singularity depends on the trace

$$TraceJX_{\eta}(N^*, P^*) = -N^*(\frac{N^{*2} + 2N^*B - 2N^*A + B^2 - BA + A}{N^* + B - A})$$

If  $A = \frac{(1+B)(C-D-\overline{C})}{C-D}$  the Trace is zero and  $(N^*, P^*)$  is center.

### 4 Conclusions

Refuge capacity determines the number of equilibrium points. The stability of the equilibrium point into the first quadrant model Rosenzweig-Mac Arthur depends on the size of the refuge, showing a different conclusion in the previous work results in the Refuge stabilizes predator interactions. When the capacity of the refuge is low in the studied model you have stability in the equilibrium point into the first quadrant. Predators can be extinguished when the size of the refuge is great. When you have a refuge to protect a number of prey predation they do not always have the conditions for coexistence of predator and prey such conditions can depend on the size of the refuge, getting a different result at [3]. We show limit cycles are obtained, which indicate periodic solutions, resulting in oscillatory behavior in populations.

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