On Connected Closed Geodetic Numbers of Some Graphs

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Abstract
This study deals primarily with the connected closed geodetic numbers of some graphs, a closed geodetic closure invariant introduced by Buckley and Harary [2]. For $S \subseteq V(G)$, where $G$ is a connected graph, the geodetic closure $I_G[S]$ of $S$ is the set of all vertices lying on some $u$-$v$ geodesic where $u$ and $v$ are in $S$. In this paper, select vertices of $G$ sequentially as follows:

Select a vertex $v_1$ and let $S_1 = \{v_1\}$. Select a vertex $v_2 \neq v_1$ and let $S_2 = \{v_1, v_2\}$, then determine $I_G[S_2]$. If $I_G[S_2] \neq V(G)$, then successively select vertex $v_i \notin I_G[S_{i-1}]$ and let $S_i = \{v_1, v_2, ..., v_i\}$ for $i = 1, 2, ..., k$. Then determine $I_G[S_k]$.

The connected closed geodetic number of a graph, denoted by $ccgn(G)$, is defined to be the smallest $k$ whose selection of $v_k$ in the given manner yields $I_G[S_k] = V(G)$, where $\langle S \rangle$ is connected. In this paper, the connected closed geodetic numbers of some graphs and the join of some connected graphs $G$ and $H$ were determined.
1 Introduction

The concept on connected closed geodetic numbers of graphs follows from the definition of a closed geodetic numbers of graphs which is introduced by Buckley and Harary [2]. The closed geodetic numbers of graphs is studied by Aniversario et.al in [1]. The idea evolved from two classes of graphical games called achievement and avoidance games.

The concept involves closed geodetic closure of a set $S \subseteq V(G)$ of a graph $G$ denoted by $I_G[S]$ which is the set of all vertices on geodesics which is the shortest path between two vertices in $S$. The achievement and avoidance game is modified for the purpose of the closed geodetic concept and goes like this. The first player $A$ chooses a vertex $v_1$ of $V(G)$. The second player $B$ then selects $v_2 \neq v_1$ and determines $I_G[S_2]$ for $S_2 = \{v_1, v_2\}$. If $I_G[S_2] \neq V(G)$, then $A$ picks $v_3 \notin I_G[S_2]$ for $S_3 = \{v_1, v_2, v_3\}$. In general, $A$ and $B$ alternately select a new vertex in this manner. The first player who selects a vertex $v_k \notin I_G[S_{k-1}]$ such that $I_G[S_k] = V(G)$ for $S_k = \{v_1, v_2, \ldots, v_k\}$, wins the achievement game; in the avoidance game he is the loser.

The study on closed geodetic numbers leads us to study closely on connected closed geodetic numbers. The researchers find it interesting to study the set $S \subseteq V(G)$ for connected graphs that gives the connected closed geodetic number of $G$ where $\langle S \rangle$ is connected. Most of the results are parallel to the results in [1].

2 Preliminary Concepts and Results

Definition 2.1 For every two vertices $u$ and $v$ of $G$, the symbol $I_G[u, v]$, where $u, v \in V(G)$, is used to denote the interval containing $u, v$ and all vertices lying in some $u$-$v$ geodesic. A subset $S$ of $V(G)$ is a geodetic cover of $G$ if $I_G[S] = V(G)$, where $I_G[S] = \bigcup_{u,v \in S} I_G[u, v]$. $I_G[S]$ is called the geodetic closure of $S$ in $G$.

Definition 2.2 Given a connected graph $G$ and $S \subseteq V(G)$, the set $S$ is a closed geodetic cover of $G$ if $S = \{v_1, v_2, \ldots, v_k\}$ and is obtained by choosing the vertices $v_1, v_2, \ldots, v_k$ such that the following hold:

(i) $v_1 \neq v_2$;

(ii) $v_i \notin I_G[S_{i-1}]$ for $3 \leq i \leq k$; and

(iii) $I_G[S_k] = V(G)$,
where \( S_i = \{v_1, v_2, \ldots, v_i\} \) for all \( i = 1, 2, \ldots, k \). The collection of all closed geodetic covers of \( G \) is denoted by \( C^*(G) \). The \textit{closed geodetic number} of \( G \), is given by
\[
cgn(G) = \min \{|S| : S \in C^*(G)\}.
\]
A set \( S \in C^*(G) \) with \(|S| = cgn(G)\) is called the \textit{closed geodetic basis} of \( G \), denoted by \( cgb(G) \).

\textbf{Definition 2.3} The \textit{geodetic number}, \( gn(G) \) of a graph \( G \) is the minimum cardinality among all geodetic covers of \( G \), that is,
\[
 gn(G) = \min\{|S| : S \subseteq V(G)\}
\]
and \( I_G[S] = V(G) \}. Furthermore, a geodetic cover of smallest cardinality is called a \textit{geodetic basis} of \( G \).

\textbf{Definition 2.4} Let \( G \) be a connected graph of order \( n \). Let \( S \subseteq V(G) \) such that \( S \in C^*(G) \). Then the \textit{connected closed geodetic number} of a graph \( G \), denoted by \( ccgn(G) \) is given by
\[
 ccgn(G) = \min\{|S| : \langle S \rangle \text{ is connected}\}.
\]

\textbf{Example 2.5} Consider the graphs in Figure 1 below.

\( C_5 \):

\( \langle S_3 \rangle \):

\( \langle S_4 \rangle \):

\( \langle S_5 \rangle \):

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{The cycle \( C_5 \), and its subgraphs \( \langle S_3 \rangle \), \( \langle S_4 \rangle \), and \( \langle S_5 \rangle \).
}
\end{figure}

Let \( S_1 = \{v_1\}, S_2 = \{v_1, v_2\}, \text{ and } S_3 = \{v_1, v_2, v_4\} \}. Note that \( I_G[S_3] = I_G[v_1, v_2] \cup I_G[v_1, v_4] \cup I_G[v_2, v_4] = \{v_1, v_2\} \cup \{v_1, v_3, v_4\} \cup \{v_2, v_3, v_4\} = \{v_1, v_2, v_3, v_4, v_5\} = V(G) \). However, \( \langle S_3 \rangle \) is not connected as shown in Figure 1 and since it does not satisfy the definition of connected closed geodetic number, we need to choose another vertex in order for \( \langle S \rangle \) to be connected and
$I_G[S] = V(G)$. Now, select $v_3 \notin S_3$ that gives $S_4 = S_3 \cup \{v_3\} = \{v_1, v_2, v_3, v_4\}$. Note that


$$= \{v_1, v_2\} \cup \{v_1, v_2, v_3\} \cup \{v_1, v_5, v_4\} \cup \{v_2, v_3\} \cup \{v_2, v_3, v_4\} \cup \{v_3, v_4\}$$

$$= \{v_1, v_2, v_3, v_4, v_5\}$$

$$= V(G)$$

and $\langle S_4 \rangle$ is connected as shown in Figure 1. Similarly, we can also select $v_5 \notin S_3$ which gives $S_5 = S_3 \cup \{v_5\} = \{v_1, v_2, v_4, v_5\}$ where

$$I_G[S_5] = I_G[v_1, v_2] \cup I_G[v_1, v_4] \cup I_G[v_1, v_5] \cup I_G[v_2, v_4] \cup I_G[v_2, v_5] \cup I_G[v_3, v_5]$$

$$= \{v_1, v_2\} \cup \{v_1, v_5, v_4\} \cup \{v_1, v_5\} \cup \{v_2, v_3, v_4\} \cup \{v_2, v_1, v_5\} \cup \{v_4, v_5\}$$

$$= \{v_1, v_2, v_3, v_4, v_5\}$$

$$= V(G)$$

and note that $\langle S_5 \rangle$ is also connected as shown in Figure 1. Then it follows that, $ccgn(C_5) = 4$.

**Remark 2.6** For any connected graph $G$, $cgn(G) \leq ccgn(G)$. That is, every connected closed geodetic cover is a closed geodetic cover.

For the graph given in Figure 1, $S_4$ and $S_5$ are two different connected closed geodetic covers of $G$. Thus the following result is immediate.

**Remark 2.7** There can be more than one connected closed geodetic covers of $G$.

**Theorem 2.8** For any connected nontrivial graph $G$ of order $n$,

$$2 \leq cgn(G) \leq ccgn(G) \leq n.$$  

**Proof:** Any closed geodetic set needs at least two vertices and therefore, $cgn(G) \geq 2$. Let $S$ be any connected geodetic set of $G$ with minimum cardinality. By Theorem 2.6, it follows that $2 \leq cgn(G) \leq ccgn(G)$. Since $V(G)$ induces a connected closed geodetic set of $G$, it implies that $ccgn(G) \leq n$. Thus, $2 \leq cgn(G) \leq ccgn(G) \leq n$. $\blacksquare$

**Theorem 2.9** Let $G = C_n$. Then for $n \geq 3$,

$$ccgn(G) = \begin{cases} \left\lfloor \frac{n-1}{2} \right\rfloor + 2, & \text{if } n \text{ is odd}, \\ \frac{n}{2} + 1, & \text{if } n \text{ is even}. \end{cases}$$

**Proof:** Let $G = C_n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Consider the following cases for the order $n$ of graph $G$. 
Case 1: When \( n \) is odd, say \( n = 2k + 1 \), for some integer \( k \). Let \( S_1 = \{v_1\} \) and \( S_2 = \{v_1, v_n\} \) where \( I_G[S_2] = \{v_1, v_n\} \). Thus \( v_2 \notin I_G[S_2] \) so that \( S_3 = S_2 \cup \{v_2\} \) where \( I_G[S_3] = \{v_1, v_n, v_2\} = \{v_1, v_n\} \cup \{v_2\} \neq V(G) \). Now, for the cycle \( C_n \) of order \( n = 2k + 1 \), it is the union of two paths \([v_{2k+1}, v_1, v_2, \ldots, v_{k+1}]\) and \([v_{k+1}, v_{k+2}, \ldots, v_k, \ldots, v_{2k}, v_{2k+1}]\) of lengths \( k + 1 \) and \( k \), respectively. Let \( S = \{v_{2k+1}, v_1, v_2, \ldots, v_{k+1}\} \). Observe that \( I_G[S] \) contains the vertices on \([v_{k+1}, \ldots, v_{2k+1}]\) where \( \{v_{k+1}, \ldots, v_{2k+1}\} = I_G[v_{k+1}, v_{2k+1}] \). Thus, \( I_G[S] = V(G) \). Hence, the minimum number of vertices for \( S \in C^*(G) \) for which \( \langle S \rangle \) is connected is attained if

\[
\{v_1, v_{n=2k+1}, v_2, \ldots, v_{k=\frac{n-1}{2}}, v_{k+1=\frac{n+1}{2}}\} = \{v_1, v_2, \ldots, v_k, v_{k+1}\} \cup \{v_{n=2k+1}\}
\]

and \( \langle S \rangle \) is connected. Therefore,

\[
|S| = |\{v_1, v_2, \ldots, v_k, v_{k+1}\} \cup \{v_{n=2k+1}\}|
= (k + 1) + 1
= \left(\frac{n-1}{2}\right) + 1 + 1
= \left(\frac{n-1}{2}\right) + 2.
\]

Thus, \( ccgn(G) = \left[\frac{n-1}{2}\right] + 2 \).

Case 2: When \( n \) is even, say \( n = 2k \) for some integer \( k \). Let \( S_1 = \{v_1\} \) and \( S_2 = \{v_1, v_{n=2k}\}\) where \( I_G[S_2] = \{v_1, v_{n=2k}\} \). Thus, \( v_2 \notin I_G[S_2] \) so that \( S_3 = S_2 \cup \{v_2\} = \{v_1, v_{n=2k}, v_2\} \), where \( I_G[S_3] = \{v_1, v_{n=2k}, v_2\} = S_3 \). Hence, for any \( i = 1, 2, \ldots, \frac{n}{2} - 1 = k - 1, v_i \notin I_G[S_{i-1}] \) and \( I_G[S_i] = \{v_1, v_{n=2k}, v_2, \ldots, v_{i=k-1}\} \neq V(G) \) for \( S_i = \{v_1, v_{n=2k}, v_2, \ldots, v_{i=k-1}\} \).

Now, for \( i = \frac{n}{2} = k \), the length of the path \([v_{2k}, v_{2k-1}, \ldots, v_k] \) is the same with the length of the path \([v_{2k}, v_1, v_2, \ldots, v_k] \). Hence, \( I_G[v_{2k}, v_k] = V(G) \).

That is, the minimum number of vertices for \( S \in C^*(G) \) where \( \langle S \rangle \) is connected is attained if \( S = S_k \cup \{v_n\} \), \( S_k = \{v_1, v_2, v_{n=2k}\} \). Hence, \( ccgn(G) = |S_k \cup \{v_n\}| = |S_k| + 1 = k + 1 = \frac{n}{2} + 1 \), when \( n \) is even.

Theorem 2.10 Let \( G = P_n \). Then \( ccgn(G) = n \), for all \( n \geq 2 \).
Proof: Let \( G = P_n \) and \( V(G) = \{v_1, v_2, v_3, \ldots, v_n\} \) as shown in Figure 2.

![Figure 2: A path \( P_n \) of order \( n \), for all \( n \geq 2 \)](image)

Let \( S_1 = \{v_1\} \) and \( S_2 = \{v_1, v_2\} \). Then \( I_G[v_1, v_2] = \{v_1, v_2\} = S_2 \). Let, \( v_3 \notin I_G[S_2] \) which gives \( S_3 = S_2 \cup \{v_3\} = \{v_1, v_2, v_3\} \), where \( I_G[S_3] = \{v_1, v_2, v_3\} = S_3 \). Hence, for any \( i = 1, 2, \ldots, k < n, I_G[S_i] = S_{i-1} \cup \{v_i\} = S_i \) and \( v_i \notin I_G[S_{i-1}] \). Therefore, the minimum number of vertices that induces a connected set is attained for \( S \in C^*(G) \) if \( S = S_{n-1} \cup \{v_n\} = \{v_1, v_2, \ldots, v_n\} = V(G) \) and \( \langle S \rangle \) is connected whose \( I_G[S] = V(G) \). Thus, \( ccgn(G) = n \) for all \( n \geq 2 \).
The following result is immediate.

**Corollary 2.11** If \( n = 2 \), then \( ccgn(P_n) = cgn(P_n) \).

**Theorem 2.12** Let \( G \) be a connected graph of order \( p \). Then \( ccgn(G) = p \) if and only if \( G = K_p \).

*Proof:* Let \( G = K_p \). For a complete graph \( G \) and any \( u, v \in V(G) \), \( I_G[u, v] = \{u, v\} \). Thus, \( V(G) \) is the only set of vertices of \( G \) for which \( I_G[V(G)] = V(G) \), where \( \langle V(G) \rangle \) is connected. Hence, \( ccgn(G) = |V(G)| = p \). Conversely, suppose that \( G \neq K_p \). Then there exist \( u \) and \( v \) in \( V(G) \) such that \( d_G(u, v) = 2 \). We construct a set of vertices in \( G \), \( S = \{v_1, v_2, \ldots, v_k\} \in C^*(G) \), where \( k < p \) such that \( v_1 = u \) and \( v_2 = v \). Since \( I_G[v_1, v_2] \neq \{v_1, v_2\} \), we have \( I_G[S] \neq S \). In fact, \( I_G[S] = V(G) \). Consequently, \( k < p \). This means that \( ccgn(G) < p \), a contradiction to the assumption. Thus, \( G = K_p \).

**Theorem 2.13** Let \( G \) be a connected graph of order \( p \). Then, \( cgn(G) = ccgn(G) \) if and only if \( G = K_p \), a complete graph of order \( p \geq 2 \).

*Proof:* Let \( G = K_p \) be a complete graph of order \( p \). Then by Theorem 2.12, \( cgn(G) = p = ccgn(G) \). Conversely, let \( cgn(G) = ccgn(G) \). Suppose that \( G \neq K_p \). Then there exist \( u, v \in V(G) \) such that \( d_G(u, v) = 2 \). Let \([u, w, v] \) be a \( u \)-\( v \) geodesic. Then we can construct a set \( S = \{v_1, v_2, \ldots, v_k\} = V(G)\setminus\{w\} \), since \( w \in I_G[u, v] \) for which \( v_1 = u \) and \( v_k = v \) and \( I_G[S] = V(G) \). Since \( \langle S \rangle \) is connected, and \( k < p \), we have \( ccgn(G) < p \), a contradiction to the assumption that \( ccgn(G) = p \). Therefore, \( G = K_p \).

**Theorem 2.14** Let \( G = K_{m,n} \). Then \( ccgn(G) = \min\{m, n\} + 1 \), for \( m, n > 2 \).

*Proof:* Let \( G = K_{m,n} \) and let \( U \) and \( W \) be the partite sets of \( V(G) \). Note that the only closed geodetic covers of \( K_{m,n} \) are \( U, W, U \cup \{w\} \) for some \( w \in W \), and \( W \cup \{u\} \) for some \( u \in U \) with \( |U| = m \) and \( |V| = n \), respectively. Since only \( U \cup \{w\} \) and \( W \cup \{u\} \) are the connected closed geodetic covers of \( G \), we have

\[
ccgn(G) = \min\{|U| + 1, |V| + 1\} \\
= \min\{m + 1, n + 1\} \\
= \min\{m, n\} + 1.
\]

**Corollary 2.15** If \( m = n \), then \( ccgn(K_{m,n}) = n + 1 \), for \( n \geq 2 \).

*Proof:* Let \( G = K_{m,n} \) and \( m = n \) where \( n \geq 2 \). Then by Theorem 2.14,

\[
ccgn(G) = \min\{m, n\} + 1 \\
= \min\{n, n\} + 1 \\
= n + 1.
\]
The following known results on the join of some connected graphs are found in [1].

**Theorem 2.16** Let $H$ be a connected noncomplete graph and $G = H + K_p$. If $S$ is a closed geodetic basis of $G$, then $S \subseteq V(H)$ and $S$ is a 2-path closure absorbing set in $H$.

**Corollary 2.17** If $H$ is a connected noncomplete graph and $G = H + K_p$, then $cgn(H + K_p) = \min\{|S| : S \subseteq V(H), S \in C^*(G) \text{ and } P_2[S]_H = V(H)\}$.

**Corollary 2.18** If $H$ is a connected noncomplete graph and $\text{diam}(H) = 2$, then $cgn(H + K_p) = cgn(H)$.

**Theorem 2.19** Let $G = H + K$, where $H$ and $K$ are connected and noncomplete graphs. If either

(i) $S \subseteq V(H)$, $P_2[S]_H = V(H)$ and $S \in C^*(H)$ or

(ii) $S \subseteq V(K)$, $P_2[S]_K = V(K)$ and $S \in C^*(K)$,

then $S \in C^*(G)$.

**Theorem 2.20** Let $G = H + K$, where $H$ and $K$ are connected and noncomplete graphs. If $S$ is a closed geodetic basis of $G$, then either

(i) $S \subseteq V(H)$, $P_2[S]_H = V(H)$ or

(ii) $S \subseteq V(K)$, $P_2[S]_K = V(K)$.

**Corollary 2.21** $cgn(P_n + K_p) = \left\lceil \frac{n+1}{2} \right\rceil$, for $n \geq 3$.

**Corollary 2.22** $cgn(C_n + K_p) = \left\lceil \frac{n}{2} \right\rceil$, for $n \geq 4$.

### 3 Connected Closed Geodetic Numbers of the Join of Some Graphs

In this section, we discuss the connected closed geodetic number of a graph $G$ obtained from the join of two graphs $G$ and $H$.

**Example 3.1** Consider the graphs $P_3$, $C_4$, and $P_3 + C_4$ in Figure 4.

Let $S = \{b, a, e\}$. Then $I_G[S] = I_G[b, a] \cup I_G[b, e] \cup I_G[a, e] = \{b, a\} \cup \{b, a, g, f, c, d, e\} \cup \{a, e\} = \{b, a, g, f, c, d, e\} = V(G)$. Thus, $I_G[S] = V(G)$. Further, $\langle S \rangle$ is connected as shown in Figure 5. Then it follows that $ccgn(P_3 + C_4) = 3$. 
Figure 3: The graphs $P_3$, $C_4$, and $P_3 + C_4$

Figure 4: The graph $\langle S \rangle$, subgraph of $P_3 + C_4$
Theorem 3.2 If \( H \) is a connected noncomplete graph and \( G = K_p + H \), then \( ccgn(G) = \min \{|S| : S = T \cup \{v\} \text{ where } v \in V(K_p) \text{ and } P_2[T]_H = V(H)\} \).

Proof: Let \( G = K_p + H \) where \( H \) is a connected nocomplete graph. Let \( T \) be the set of minimum cardinality such that \( T \in C^*(G) \). Then by Theorem 2.19, \( P_2[T]_H = V(H) \) for which \( T \subseteq V(H) \). Thus, \( \langle T \rangle \) is not connected in \( G \). Hence, construct \( S = T \cup \{v\} \) where \( v \in V(K_p) \). Thus, \( \langle S \rangle \) is connected in \( G \). Therefore, the result follows.

Corollary 3.3 If \( H \) is a connected noncomplete graph and \( G = K_p + H \), then \( ccgn(G) = cgn(G) + 1 \).

Proof: Let \( G = K_p + H \). By Theorem 3.2 and Corollary 2.16, \( ccgn(G) = cgn(G) + 1 \).

Corollary 3.4 Let \( G = K_p + P_n \). Then \( ccgn(G) = \left\lceil \frac{n+1}{2} \right\rceil + 1 \), for \( n \geq 3 \).

Proof: Let \( G = K_p + P_n \). By Theorem 3.3 and Corollary 2.21, we have

\[
ccgn(G) = ccgn(K_p + P_n) = cgn(P_n) + 1 = \left\lceil \frac{n+1}{2} \right\rceil + 1
\]

Remark 3.5 For \( n = 1,2 \), \( ccgn(K_p + P_1) = cgn(K_p + P_1) = p + 1 \) and \( ccgn(K_p + P_2) = cgn(K_p + P_2) = p + 2 \).

Corollary 3.6 Let \( G = K_p + C_n \). Then \( ccgn(G) = \left\lceil \frac{n}{2} \right\rceil + 1 \), for \( n \geq 4 \).

Proof: Let \( G = K_p + C_n \). By Corollary 3.3 and Corollary 2.22, we have

\[
ccgn(G) = cgn(G) + 1 = cgn(K_p + C_n) = cgn(C_n) + 1 = \left\lceil \frac{n}{2} \right\rceil + 1
\]

Corollary 3.7 For the wheel \( W_n = C_n + K_1 \) of order \( n+1 \), we have \( ccgn(W_n) = \left\lceil \frac{n}{2} \right\rceil + 1 \) for \( n \geq 3 \).

Corollary 3.8 If \( H \) is a connected noncomplete graph and \( diam(H) = 2 \), then \( ccgn(H + K_p) = cgn(H) + 1 \).

Proof: Let \( G = H + K_p, S \subseteq V(H) \) such that \( S \in C^*(H) \) where \( diam(H) = 2 \). That is, for every \( u,v \in V(H) \), \( d_H(u,v) \leq 2 \). Thus \( P_2[S]_H = V(H) \). However, \( \langle S \rangle \) is not connected but \( S = cgb(H) \) by Corollary 2.18. Thus, by Theorem 2.19, \( S = cgb(G) \). Consider the set \( T = S \cup \{v\} \) where \( v \in V(K_p) \). Then \( \langle T \rangle \) is connected. Note that \( T \) can be constructed so that \( T \in C^*(G) \) by letting \( v_1 \in V(H) \) and choose \( v_2 \in V(K_p) \). Since \( S = cgb(G) \), \( |S| \) is minimum. Thus, \( T \in C^*(G) \) has minimum cardinality for which \( \langle T \rangle \) is connected. Therefore, \( ccgn(H + K_p) = cgn(H) + 1 \).
Corollary 3.9  Let $G = K_{m,n} + K_p$. Then $ccgn(G) = \min\{m, n\} + 1$, for all $m, n \geq 2$.

Proof: Let $H = K_{m,n}$. Then by Corollary 3.8 and Theorem 2.14, we have $ccgn(K_{m,n} + K_p) = cgn(K_{m,n}) + 1 = \min\{m, n\} + 1$. ■

References


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