Some Relations Involving Hypergeometric Functions of Three and Four Variables

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Abstract

The aim of this paper is to derive certain relations involving Exton’s functions $K_1$ and $K_2$, generalized Horn’s function $(k)H_k^{(4)}$ and Lauricella’s function $F_C^{(4)}$. These relations are derived with the help of the Laplace integral representations of Exton’s quadruple hypergeometric functions. Some deductions from these relations lead us to a number of (new and known) reduction formulas of Lauricella’s functions $F_A^{(3)}$ and $F_C^{(3)}$ and Exton’s functions $X_2, X_4$ and $X_8$.

Keywords: Exton’s functions, Generalized Horn’s function, Lauricella’s functions, Laplace integral

1. Introduction

Exton [1; p. 78-82] gave the definitions and the Laplace integral representations of the quadruple hypergeometric functions $K_1$ and $K_2$ as follows:

$$K_1(a,a,a;a;b,b;c_2,d_2,d_4;x,y,z,t)$$
\[
= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_{m+n} (c_1)_{p} (c_2)_{q} n^m y^n z^p t^q}{(d_1)_m (d_2)_n (d_3)_p (d_4)_q} m! n! p! q!
\]

(1.1)

\[
= \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-u} u^{a-1} \Psi_2(b; d_1, d_2; xu, yu) F_1(c_1; d_3; zu) F_1(c_2; d_4; tu) du
\]

(1.2)

\[K_{13}(a, a, a, a; b_1, b_2, b_3, b_4; c, c, d_1, d_2; x, y, z, t)
\]

\[
= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b_1)_{m} (b_2)_{n} (b_3)_{p} (b_4)_{q} n^m y^n z^p t^q}{(c)_{m+n} (d_1)_p (d_2)_q} m! n! p! q!
\]

(1.3)

\[
= \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-u} u^{a-1} \Phi_2(b_1, b_2; c; xu, yu) F_1(b_3; d_1; zu) F_1(b_4; d_2; tu) du
\]

(1.4)

Where \( F_1 \) is Kummer’s function \([7; p. 36]\) and the functions \( \Phi_2, \Psi_2 \) are the confluent hypergeometric functions of two variables \([7; p. (58, 59)]\).

The generalized Horn’s function of four variables \((k)H_4^{(4)}\) is defined by Exton \([1; p.97]\) as follows:

\[\begin{align*}
(k)H_4^{(4)}(a, b_1, \ldots, b_4; c_1, \ldots, c_4; x_1, \ldots, x_4, x_{k+1}, \ldots, x_4)
&= \sum_{m_1, \ldots, m_4=0}^{\infty} \frac{(a)_{2m_1+\cdots+2m_4} (b_1)_{m_1} \cdots (b_4)_{m_4} x_1^{m_1} \cdots x_4^{m_4}}{(c_1)_{m_1} \cdots (c_4)_{m_4}} m_1! \cdots m_4!
\end{align*}\]

(1.5)

The Lauricella’s function of four variables \(F_C^{(4)}\) \([7; p.60]\) is defined by

\[\begin{align*}
F_C^{(4)}(a, b; c_1, c_2, c_3, c_4; x, y, z, t)
&= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_{m+n+p+q} x^m y^n z^p t^q}{(c_1)_m (c_2)_n (c_3)_p (c_4)_q} m! n! p! q!
\end{align*}\]

(1.6)

2. Main Results

The following formulas will be established in this section:

\[
\sum_{n=0}^{\infty} \frac{(a)_{n} (b)_{n} w^n}{(c)_{n} n!} K_{10}(a + n, a + n, a + n, a + n; b_1, b_1, c_1, c_2; b_1, b_1, 2c_1, 2c_2; x, y, 2z, 2t)
\]
\[ X^{-\alpha} H_4^{(3)} \left[ a, b : b_1, c_1 + \frac{1}{2}, c_2 + \frac{1}{2}, c \right] ; \frac{xy}{X^2}, \frac{z^2}{4X^2}, \frac{t^2}{4X^2}, \frac{w}{X} \]  
\[ (2.1) \]

\[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n w^n}{(c)_n n!} K_{10} (a + n, a + n, a + n, a + n; b_1, b_1, c_1, c_1; b_1, b_1, 2c_1, 2c_1; x, x, z, 2t) \]

\[ = (1 - z - t)^{-\alpha} H_4^{(4)} \left[ a, b, b_1 - \frac{1}{2}, c_1 + \frac{1}{2}, c_2 + \frac{1}{2}, c ; 2b_1 - 1 \right] \]
\[ \frac{z^2}{4(1 - z - t)^2}, \frac{t^2}{4(1 - z - t)^2}, \frac{w}{1 - z - t}, \frac{4x}{1 - z - t} \]  
\[ (2.2) \]

\[ \sum_{n=0}^{\infty} \frac{(a/2)_n ((a+1)/2)_n w^n}{(d)_n n!} K_{10} (a + 2n, a + 2n, a + 2n, a + 2n; b_1, b_1, c_1, c_1; b_1, b_1, 2c_1, 2c_1; x, y, z, 2t) \]

\[ = X^{-\alpha} F_4^{(4)} \left[ a, a + \frac{1}{2} : b_1, c_1 + \frac{1}{2}, c_2 + \frac{1}{2}, d \right] ; \frac{4xy}{X^2}, \frac{z^2}{X^2}, \frac{t^2}{X^2}, \frac{w}{X^2} \]  
\[ (2.3) \]

\[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n w^n}{(c)_n n!} K_{13} (a + n, a + n, a + n, a + n; b_1, b_1, c_1, c_1; b_1, b_1, 2c_1, 2c_1; x, x, z, 2t) \]

\[ = (1 - y - z - t)^{-\alpha} \]

\[ (2) H_4^{(4)} \left[ a, b, b_1 ; c_1 + \frac{1}{2}, c_2 + \frac{1}{2}, c , 2b_1 \right] ; \frac{z^2}{4(1 - y - z - t)^2}, \frac{t^2}{4(1 - y - z - t)^2}, \frac{w}{1 - y - z - t}, \frac{x - y}{1 - y - z - t} \]  
\[ (2.4) \]

\[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n w^n}{(c)_n n!} K_{13} (a + n, a + n, a + n, a + n; b_1, b_1, c_1, c_1; b_1, b_1, 2c_1, 2c_1; 2x, 2y, 2z, 2t) \]

\[ = X^{-\alpha} H_4^{(3)} \left[ a, b ; b_1 + \frac{1}{2}, c_1 + \frac{1}{2}, c_2 + \frac{1}{2}, c \right] ; \frac{(x - y)^2}{4X^2}, \frac{z^2}{4X^2}, \frac{t^2}{4X^2}, \frac{w}{X} \]  
\[ (2.5) \]

\[ \sum_{n=0}^{\infty} \frac{(a/2)_n ((a+1)/2)_n w^n}{(d)_n n!} \]
\[ K_{13}(a+2n,a+2n,a+2n,a+2n; b_1,b_1,c_1,c_1; 2b_1,2b_1,2c_1,2c_2; 2x,2y,2z,2t) \]
\[ = X^{-a} F_c^{(4)} \left[ \frac{a}{2}, \frac{a+1}{2}; b_1+\frac{1}{2}, c_1+\frac{1}{2}, c_2+\frac{1}{2}, d; \frac{(x-y)^2}{X^2}, \frac{z^2}{X^2}, \frac{t^2}{X^2}, \frac{w}{X^2} \right], \]

where \( X = 1 - x - y - z - t \).

The following results will be required in the proofs ([6; p.17 and p.322]), [2; p.98]):

\[ \Psi_2 [a;a,a;x,y] = e^{x+y} F_1 \left[ -; a; xy \right] \]  \hfill (2.7)

\[ \Phi_2 [a,a;2a; x,y] = e^{x} F_1 \left[ a; 2a; x - y \right] \]  \hfill (2.8)

\[ t F_1 \left[ \frac{a}{2a}; x \right] = e^{\frac{x}{2}} F_1 \left[ -; a+\frac{1}{2}; \frac{x^2}{16} \right] \]  \hfill (2.9)

\[ \int_0^\infty e^{-xu} u^{-1} F_1 \left[ -; d_1; xu^2 \right] \eta F_1 \left[ -; d_2; yu^2 \right] du \]
\[ = \frac{\Gamma(a)}{s^a} F_4 \left[ \frac{a}{2}, \frac{a+1}{2}; d_1, d_2; \frac{4x}{s^2}, \frac{4y}{s^2} \right], \]  \hfill (2.10)

where \( F_4 \) is Appell’s function [7;p.53].

\[ (\lambda)_2 = 2^{2n} \left( \frac{1}{2} \frac{1}{n} \right) \alpha \left( \frac{1}{2} \frac{1}{n} \right) \alpha, \quad n=0,1,2,\ldots \]  \hfill (2.11)

**Proofs:**

To prove (2.1), we proceed as follows: Let us denote the left hand side of (2.1) by \( S \), replace \( K_{10} \) by its integral representation and then by using the results (2.7) and (2.9), we get

\[ S = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} w^n \]
\[ = \int_0^\infty e^{-(1-x-y-z-t)u} u^{a+n-1} F_1 \left[ -; b_1; xu^2 \right] \eta F_1 \left[ -; c_1 + \frac{1}{2}; \frac{z u^2}{4} \right] \eta F_1 \left[ -; c_1 + \frac{1}{2}; \frac{t u^2}{4} \right] du \]
Now, expressing the first \( \binom{1}{n} F_1 \) into power series and using (2.10), we get

\[
S = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n}{(c)_m} \frac{w^n}{n!} \Gamma(a + n + 2m) \Gamma(a + n + 2m + 1) \Gamma(a + n) X^{a+n+2m}
\]

\[
\times F_4 \left[ \frac{a + n + 2m}{2}, \frac{a + n + 2m + 1}{2}, c_1 + \frac{1}{2}, c_2 + \frac{1}{2}, \frac{z^2}{X^2}, \frac{t^2}{X^2} \right]
\]

Expressing Appell’s function \( F_4 \) as double series and using (2.11), we get

\[
S = X^{-a} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(a)_{2m+2r+2s+t}}{(b)_m} \frac{x^m y^r z^t w^s}{m! s! t! n!} \]

\[
(b)_m (c_1 + \frac{1}{2}), (c_2 + \frac{1}{2})_s, m! s! n!
\]

This complete the proof of (2.1). The proofs of (2.2) and (2.3) are similarly. Formulas (2.4) to (2.6) are similarly established on replacing \( K_{13} \) by its integral representation and then by applying the results (2.8), (2.9), (2.10) and (2.11) during the proofs.

In (2.1), putting \( y = x \) and comparing with the result (2.2), we get

\[
(3) H^{(4)}_4 \left[ \frac{a, b, b_1, b_2, b_3, c; \quad z^2 \quad t^2 \quad w}{(1 - 2x - z - t)^2, 4(1 - 2x - z - t)^2, 4(1 - 2x - z - t)^2, 1 - 2x - z - t} \right]
\]

\[
= (1 - 2x - z - t)^{a-1} (1 - z - t)^{-a}
\]

\[
(2) H^{(4)}_4 \left[ \frac{a, b, b_1 - \frac{1}{2}, b_2, b_3, c, 2b_1 - 1; \quad z^2 \quad t^2 \quad w}{4(1 - z - t)^2, 4(1 - z - t)^2, 1 - z - t, 1 - z - t} \right], \quad (2.12)
\]

where \( b_2 = c_1 + \frac{1}{2} \) and \( b_3 = c_2 + \frac{1}{2} \).

Next, in (2.4) replacing \( x \) and \( y \) by \( 2x \) and \( 2y \) respectively and comparing with the result (2.5), we get

\[
(2) H^{(4)}_4 \left[ \frac{a, b, b_1; b_2, b_3, c, 2b_1; \quad z^2}{4Y^2}, \frac{t^2}{4Y^2}, \frac{w}{Y}, \frac{2(x - y)}{Y} \right]
\]
\[ X^{-a} Y^{a} \binom{3}{H_2} H_4 \left[ a, b ; b_1 + \frac{1}{2}, b_2, b_3, c ; \frac{(x-y)^2}{4X^2}, \frac{z^2}{4X^2}, \frac{t^2}{4X^2}, \frac{w}{X} \right], \] \hspace{1cm} (2.13)

where \( X = 1 - x - y - z - t, Y = 1 - 2y - z - t, b_2 = c_1 + \frac{1}{2} \) and \( b_3 = c_2 + \frac{1}{2} \).

### 3. Special Cases

In this section we deduce some (new and known) reduction formulas for Lauricella’s functions \( F_A^{(3)} \) and \( F_c^{(3)} \) of three variables [7; p.50] and Exton’s functions \( X_2, X_4 \) and \( X_6 \) of three variables [3].

Taking \( t = z = 0 \) and \( w = 4v \) in (2.3), we get

\[
X_4[a, b_1; d, b_1, b_1 ; v, x, y ] = (1 - x - z)^{-a} F_4 \left[ a, a + b_1 + \frac{1}{2}; b_1, d ; \frac{4xy}{(1-x-y)^2}, \frac{4v}{(1-x-y)^2} \right] \quad (3.1)
\]

Taking \( t = y = 0 \) in (2.5), we get

\[
F_A^{(3)} \left[ a, b_1, c_1, b_2b_1, 2c_1, c ; 2x, 2z, w \right] = (1 - x - z)^{-a} X_2 \left[ a, b ; b_1 + \frac{1}{2}, c_1 + \frac{1}{2}; \frac{x^2}{4(1-x-z)^2}, \frac{z^2}{4(1-x-z)^2}, \frac{w}{1-x-z} \right], \hspace{1cm} (3.2)
\]

which for \( x = 0 \), reduces to a known result of [5; p.42]

\[
F_2[a, c_1, b; 2c_1, c; 2z, w] = (1 - z)^{-a} H_4 \left[ a, b ; c_1 + \frac{1}{2}; \frac{z^2}{4(1-z)^2}, \frac{w}{1-z} \right], \hspace{1cm} (3.3)
\]

where \( F_2 \) and \( H_4 \) are Appell’s function [7;p.53] and Horn’s function [7; p.57].

Taking \( t = y = 0 \) and \( w = 4v \) in (2.6), we get a known result of [4; p.88]

\[
X_6[a, b, c; d, 2b, 2c ; v, 2x, 2z ] = (1 - x - z)^{-a} F_c^{(3)} \left[ a, a + b_1 + \frac{1}{2}; b_1, c + \frac{1}{2}; \frac{4v}{(1-x-z)^2}, \frac{x^2}{(1-x-z)^2}, \frac{z^2}{(1-x-z)^2} \right], \hspace{1cm} (3.4)
\]
Some relations involving hypergeometric functions

Taking \( w \to 0 \) in (2.12) and considering the definition (1.5)

\[
{H^{(3)}_c\left[ a; c_1, c_2, c_3; c_4; x_1, x_2, x_3 \right] = F^{(3)}_c\left[ \frac{a + 1}{2}; c_1, c_2, c_3; 4x_1, 4x_2, 4x_3 \right],}
\]

we get

\[
F^{(3)}_c\left[ \frac{a + 1}{2}; b_1, b_2, b_3; \frac{4x^2}{(1 - 2x - z - t)^2}, \frac{z^2}{(1 - 2x - z - t)^2}, \frac{t^2}{(1 - 2x - z - t)^2} \right]
= (1 - 2x - z - t)^{a} (1 - z - t)^{-a}
\]

Finally, taking \( z = y = 0 \) in (2.13), we get

\[
X_2\left[ a, b_1 - \frac{1}{2}, b_2, b_3, 2b_1 - 1; \frac{z^2}{4(1 - z - t)^2}, \frac{t^2}{4(1 - z - t)^2}, \frac{4x}{1 - z - t} \right]
= (1 - x - t)^{-a} (1 - t)^{a}
\]

\[
X_2\left[ a, b; b_1 + \frac{1}{2}, b_2, c; \frac{x^2}{4(1 - x - t)^2}, \frac{t^2}{4(1 - x - t)^2}, \frac{w}{1 - x - t} \right]
\]

References


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