Spanning Trees on Decorated Centered Cubic Lattices

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Abstract

In this paper we compute the number of spanning trees on the following decorated centred cubic lattices; base- centred cubic, side- centered cubic and edge- centred cubic lattices. For these lattices we also determine the asymptotic growth constant.

Keywords: Spanning trees, Asymptotic growth constant, Decorated centred cubic lattices

1 Introduction

The problem of the enumeration of the number of spanning trees on the network is considered by Kirchhoff in his analysis of electric circuits [6]. Kirchhoff showed that the spanning trees problem is related to the problem of computing the two-node
resistance of a resistor electrical network. The number of spanning trees is an important measure of reliability of a network and useful for designing electrical circuits. Spanning tree is of interest in statistical physics. It is also closely concerned with the q-state Potts model [5, 15].

There are two approaches for calculating the number of spanning trees are the matrix tree theorem (Laplacian matrix) [9, 10, 11] and the Tutte polynomial [1, 4, 7, 13]. The enumeration of spanning trees and the computation of their asymptotic growth constants on uniform lattices or graphs were studied extensively, see for instance [2, 3, 8, 12, 14].

In this paper we will use the matrix tree theorem to determine the number of spanning trees and the thermodynamic limit (asymptotic growth constant) for the following decorated cubic centered lattices: base-centered cubic, side-centered cubic and edge-centered cubic lattice.

2 Definitions and method (A brief formulation)

In this section, we briefly present basic definitions, expressions and the general method (matrix tree theorem) that we use in this work.

Consider a lattice \( L \) that is a uniform (periodic) tiling of \( d \)-dimensional space and is decomposable into a hypercubic array of \( N_1 \times N_2 \times \cdots \times N_d \) unit cells, each containing \( s \) sites labeled by 1, 2, \ldots, \( s \) so that the number of sites in the lattice is \( n = s N_1 N_2 \cdots N_d \). The unit cell can be specified by the coordinate \( n = (n_1, n_2, \ldots, n_d) \), where \( n_i = 0, 1, 2, \ldots, N_i - 1 \). The connection between the sites of the unit cells \( n \) and \( n' \) can be described by an adjacency matrix \( A(n, n') \) which is \( s \times s \) matrix and defined by

\[
A_{\alpha \beta}(n, n') = \begin{cases} 
1 & \text{if site } \alpha \text{ in cell } n \text{ and site } \beta \text{ in cell } n' \text{ are adjacent} \\
0 & \text{otherwise}
\end{cases}
\] (1)

By imposing periodic boundary conditions, the translational symmetry is \( A(n, n') = A(n - n') \) and therefore \( A(n) = a(n_1, n_2, \ldots, n_d) \). The degree matrix \( D_s \) for a unit cell is a \( s \times s \) diagonal matrix whose elements

\[
D_{\alpha \beta} = \kappa_{\alpha} \delta_{\alpha \beta}
\] (2)

where \( \kappa_{\alpha} \) is the degree or coordination number of site \( \alpha \) and \( \delta_{\alpha \beta} \) is the Kronecker delta function defined as \( \delta_{\alpha \beta} = 1 \) if \( \alpha = \beta \) and \( \delta_{\alpha \beta} = 0 \) if \( \alpha \neq \beta \). The Laplacian matrix of the lattice is defined by

\[
L(\Theta) = D_s - \sum_n A(n)e^{in\Theta}
\] (3)

where \( \Theta = (\theta_1, \theta_2, \ldots, \theta_d) \) is the \( d \)-dimensional vector. The very important theorem for counting the number of spanning trees \( N_{ST} \) (\( L \)) in graph theory is given by [1, 7]
Spanning trees on decorated centered cubic lattices

\[ N_{ST}(L) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \]  

(4)

where \( \lambda_i \) are the non-zero eigenvalues of the Laplacian matrix \( L \) of the lattice.

It is known that a determinant of the Laplacian matrix is equal to the product of its eigenvalues, so that \( N_{ST}(L) \) can be written as [8]

\[ N_{ST}(L) = \frac{\lambda_1 \lambda_2 \ldots \lambda_{n-1}}{sN_1 N_2 \ldots N_d} \prod_{i=0}^{N_1-1} \prod_{j=0}^{N_2-1} \prod_{k=0}^{N_d-1} \det L \left( \theta_1 = \frac{2 \pi \ell_1}{N_1}, \theta_2 = \frac{2 \pi \ell_2}{N_2}, \ldots, \theta_d = \frac{2 \pi \ell_d}{N_d} \right) \]  

(5)

where \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1} \) are the non-zero eigenvalues of \( L(0,0,0) \).

The number of spanning trees \( N_{ST}(L) \) grows asymptotically as \( \exp(nz_L) \) in the thermodynamic limit, \( n \to \infty \), where \( z_L \) is called the asymptotic growth constant for spanning trees in the thermodynamics limit on graphs or lattices and given by [8]

\[ z_L = \frac{1}{s} \int_{-\pi}^{\pi} \frac{d \theta_1}{2 \pi} \ldots \int_{-\pi}^{\pi} \frac{d \theta_d}{2 \pi} \log \left( \det (L(\theta_1, \ldots, \theta_d)) \right) \]  

(6)

It was shown in [3] that the asymptotic growth constant on the homeomorphic expansion of \( k \)-regular lattices with \( p \) lattice points inserted on each edge is given by

\[ z_{\text{homeomorphic expansion}} = \left( \frac{k}{2} - 1 \right) \log(p + 1) + z_{\text{regular lattice}} \]  

(7)

where \( k \) is the coordination number for a regular lattice.

3 Decorated centered cubic lattices, \( d=3 \)

In this section, the number of spanning trees and the asymptotic growth constants on the decorated centered cubic lattices are calculated.

3.1 Base-centered cubic lattice

The base-centered cubic lattice is a simple cubic lattice with extra vertices at the centers of the horizontal faces of the cube as shown in Fig.1. The unit cell is a cube containing two sites \( s = 2 \) numbered 1 and 2.
The base-centered cubic lattice.

Fig.1. The base-centered cubic lattice.

The degree matrix and adjacency matrices are

\[
D = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix}, \quad A(0,0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A(1,0,0) = A(-1,0,0)^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
A(0,1,0) = A(0,-1,0)^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A(0,0,1) = A(0,0,-1)^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
A(1,1,0) = A(-1,-1,0)^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

where \( A^T \) is the transpose of the matrix \( A \).

Therefore, using Eq.(3) the Laplacian matrix can be written as

\[
L(\theta_1, \theta_2, \theta_3) = D - A(0,0,0) - A(1,0,0)e^{i\theta_1} - A(-1,0,0)e^{-i\theta_1} - A(0,1,0)e^{i\theta_2} - A(0,0,1)e^{i\theta_2} - A(0,0,-1)e^{-i\theta_2} - A(1,1,0)e^{i(\theta_1 + \theta_2)} - A(-1,-1,0)e^{-i(\theta_1 + \theta_2)}
\]

Substituting Eq.(8) into the above equation yields

\[
L(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 10 - 2\cos \theta_1 - 2\cos \theta_2 - 2\cos \theta_3 & -(1 + e^{i\theta_1})(1 + e^{i\theta_2}) \\ -(1 + e^{-i\theta_1})(1 + e^{-i\theta_2}) & 4 \end{pmatrix}
\]

and hence the determinant is

\[
\det(L(\theta_1, \theta_2, \theta_3)) = 36 - 12\cos \theta_1 - 12\cos \theta_2 - 8\cos \theta_3 - 4\cos \theta_1 \cos \theta_2
\]
The non-zero eigenvalues of $L(0,0,0)$ is $\lambda_1 = 8$. Hence, the number of spanning trees $N_{ST}$ given by Eq. (5) becomes

$$N_{ST}(L_{\text{base cubic lattice}}) = \frac{1}{2N_1N_2N_3} \prod_{l_1=0}^{N_1} \prod_{l_2=0}^{N_2} \prod_{l_3=0}^{N_3} \left( 36 - 12\cos \frac{2\pi l_1}{N_1} - 12\cos \frac{2\pi l_2}{N_2} - 8\cos \frac{2\pi l_3}{N_3} - 4\cos \frac{2\pi l_1}{N_1} \cos \frac{2\pi l_2}{N_2} \right)$$

(11)

As an example, if $N_1 = N_2 = N_3 = 2$ then, number of spanning trees of the base-centered cubic lattice is

$$N_{ST}(L_{\text{base cubic lattice}}) = \frac{1}{2} \prod_{l_1=0}^{1} \prod_{l_2=0}^{1} \prod_{l_3=0}^{1} \left( 36 - 12\cos \pi l_1 - 12\cos \pi l_2 - 8\cos \pi l_3 - 4\cos \pi l_1 \cos \pi l_2 \right)$$

(12)

Using Eq. (6), the asymptotic growth constant for spanning trees is given by

$$z_{\text{base cubic lattice}} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_3}{2\pi} \log \left( 36 - 12\cos \theta_1 - 12\cos \theta_2 - 8\cos \theta_3 - 4\cos \theta_1 \cos \theta_2 \right)$$

(13a)

The numerical computation of (13) yields the value

$$z_{\text{base cubic lattice}} = 1.738692709 \ldots$$

(13b)

### 3.2 Side-centered cubic lattice

The side-centered cubic lattice is a simple cubic lattice with additional vertices at the centers of vertical faces as shown in Fig. 2. Each unit cell consisting of three vertices $s = 3$ labeled by 1, 2 and 3. Therefore, the degree matrix and the adjacency matrices are
Fig. 2. The side-centered cubic lattice.

$$D_3 = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A(0,0,0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A(1,0,0) = A(-1,0,0)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A(0,1,0) = A(0,-1,0)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A(0,0,1) = A(0,0,-1)^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A(1,0,1) = A(-1,0,-1)^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(0,1,1) = A(0,-1,1)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and one has

$$L(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 14 - 2 \cos \theta_1 - 2 \cos \theta_2 - 2 \cos \theta_3 & -(1 + e^{i\theta_1})(1 + e^{i\theta_2}) & -(1 + e^{i\theta_1})(1 + e^{i\theta_2}) \\ -(1 + e^{-i\theta_1})(1 + e^{-i\theta_2}) & 4 & 0 \\ -(1 + e^{-i\theta_1})(1 + e^{-i\theta_2}) & 0 & 4 \end{pmatrix}.$$
det \( L(\theta_1, \theta_2, \theta_3) = 16(12 - 3 \cos \theta_1 - 3 \cos \theta_2 - 4 \cos \theta_3 - \cos \theta_1 \cos \theta_3 - \cos \theta_2 \cos \theta_3), \) \hspace{1cm} (16)

The non-zero eigenvalues of \( L(0, 0, 0) \) are \( \lambda_1 = 12, \lambda_2 = 4 \). Thus the number of spanning trees of the side-centered cubic lattice is

\[
N_s(L_{\text{side cubic lattice}}) = \frac{48}{3N^3} \prod_{l=1}^{N-1} \frac{1}{(l,l,l,l)} \prod_{l=0}^{N-1} \frac{2}{2 \pi} \frac{2 \pi l_1}{N_1} - \frac{2 \pi l_2}{N_2} - \frac{2 \pi l_3}{N_3} - 64 \cos \frac{2 \pi l_1}{N_1} - 16 \cos \frac{2 \pi l_2}{N_2} - 16 \cos \frac{2 \pi l_3}{N_3}
\]

(17a)

As an example, we compute the number of spanning trees of a finite side-centered cubic lattice with \( N_1 = N_2 = N_3 = 2 \).

\[
N_s(L_{\text{side cubic lattice}}) = 2 \prod_{l_1=0}^{1} \prod_{l_2=0}^{1} \prod_{l_3=0}^{1} \left( 192 - 48 \cos \pi l_1 - 48 \cos \pi l_2 - 64 \cos \pi l_3 - 16 \cos \pi l_1 \cos \pi l_3 - 16 \cos \pi l_2 \cos \pi l_3 \right)
= 230 \, 897 \, 441 \, 832 \, 960
\]

(17b)

From Eq. (6) the asymptotic growth constant for spanning trees is given by

\[
z_{\text{side cubic lattice}} = \frac{1}{3} \left[ \frac{d}{d \pi} \frac{d}{d \theta_1} \frac{d}{d \theta_2} \frac{d}{d \theta_3} \right] \log \{ 16(12 - 3 \cos \theta_1 - 3 \cos \theta_2 - 4 \cos \theta_3 - \cos \theta_1 \cos \theta_3 - \cos \theta_2 \cos \theta_3) \}
\]

and the numerical evaluation gives \( z_{\text{side cubic lattice}} = 1.7211738959 \ldots \)

(18)

### 3.3 Edge-centered cubic lattice

The edge-centered cubic lattice is shown in Fig. 3, which is a homeomorphic expansion of simple cubic lattice with one vertex (site) inserted on each edge-midpoint. The unit cell is a cube containing four sites \( s = 4 \) numbered 1, 2, 3 and 4. The degree matrix and the adjacency matrices are

\[
D_4 = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, A(0,0,0) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

(19a)
Also the Laplacian matrix is given by

\[
\mathbf{L}(\theta_1, \theta_2, \theta_3) = \begin{bmatrix}
6 & -1-e^{-i\theta_1} & -1-e^{-i\theta_2} & -1-e^{-i\theta_3} \\
-1-e^{i\theta_1} & 2 & 0 & 0 \\
-1-e^{i\theta_2} & 0 & 2 & 0 \\
-1-e^{i\theta_3} & 0 & 0 & 2
\end{bmatrix},
\]

with \(\det[\mathbf{L}(\theta_1, \theta_2, \theta_3)] = 24 - 8\cos \theta_1 - 8\cos \theta_2 - 8\cos \theta_3\). The Laplacian matrix

\[
A (1,0,0) = A (-1,0,0)^T = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad (19b)
\]

\[
A (0,1,0) = A (0,-1,0)^T = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad (19c)
\]

\[
A (0,0,1) = A (0,0,-1)^T = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \quad (19d)
\]
L(0,0,0) has the eigenvalues of $\lambda_1 = 8$, $\lambda_2 = \lambda_3 = 2$. Thus the number of spanning trees of the edge-centered cubic lattice is

$$N_{ST}(L_{\text{edge cubic lattice}}) = \frac{32}{4N_1N_2N_3} \prod_{(\ell_1, \ell_2, \ell_3 \neq 0)}^{N_1-1} \prod_{\ell_i=0}^{N_3} \left( 24 - 8\cos \frac{2\pi \ell_1}{N_1} - 8\cos \frac{2\pi \ell_2}{N_2} - 8\cos \frac{2\pi \ell_3}{N_3} \right)$$

As an example, we compute the number of spanning trees of a finite edge-centered cubic lattice with $N_1 = N_2 = N_3 = 2$.

$$N_{ST}(L_{\text{edge cubic lattice}}) = \prod_{(\ell_1, \ell_2, \ell_3 \neq 0)}^{1} \left( 24 - 8\cos \pi \ell_1 - 8\cos \pi \ell_2 - 8\cos \pi \ell_3 \right)$$

$$= 503316480.$$  

The asymptotic growth constant is

$$z_{\text{edge cubic lattice}} = \frac{1}{4} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \int_{\theta_1}^{\pi} \frac{d\theta_2}{2\pi} \int_{\theta_2}^{\pi} \frac{d\theta_3}{2\pi} \log(24 - 8\cos \theta_1 - 8\cos \theta_2 - 8\cos \theta_3)$$

The numerical evaluation of (23) gives $z_{\text{edge cubic lattice}} = 0.76492094....$. Since the edge-centered cubic lattice is the homeomorphic expansion of the simple cubic lattice ($k = 6$) with $p = 1$ vertex inserted on each edge, one can use Eq. (7) to calculate the asymptotic growth constant of the edge-centered cubic lattice from that of the simple cubic lattice [8,12], we have

$$z_{\text{edge cubic lattice}} = \frac{2\log 2 + z_{\text{simple cubic lattice}}}{4} = 0.765111$$

One can note that inserting vertices on the edges of the simple cubic lattice reduce its asymptotic growth constant.

References


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