Some Integral Inequalities for Convex Functions via Riemann-Liouville Integrals

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Abstract

In this paper, by setting up a generalized integral identity for differentiable functions via Riemann-Liouville fractional integrals, the author derive new estimates on generalization of Hermite-Hadamard and Ostrowski types inequalities for functions whose derivatives in the absolute value at certain powers are convex.

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1 Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a function defined on the interval \( I \) of real numbers. Then \( f \) is called to be convex on \( I \) if the following inequality

\[
    f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)
\]

for all \( a, b \in I \) and \( t \in [0, 1] \). There are many results associated with convex functions in the area of inequalities, but some of those is the classical Hermite-Hadamard inequality and Ostrowski inequality, respectively [10, 13]:
Theorem 1.1. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). Then following double inequality holds:

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

(1)

Theorem 1.2. [11] Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( a, b \in I^0 \) with \( a < b \). If \( |f'(x)| \leq M, x \in [a,b] \), then the following inequality holds:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{2} \right].
\]

(2)

Definition 1. The beta function, also called the Euler integral of the first kind, is a special function defined by

\[
\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \ x, y > 0,
\]

and

\[
\beta(a,x,y) = \int_0^a t^{x-1}(1-t)^{y-1} \, dt, 0 < a < 1, x, y > 0,
\]

is incomplete Beta function.

Definition 2. A function \( f : [a,b] \to (0,\infty) \) is said to be \( s \)-convex in the second sense if the following inequality

\[
f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)
\]

holds for all \( a, b \in I, t \in [0,1], \) and \( s \in (0,1] \).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. Let \( f \in L([a,b]) \). The symbols \( J_{a^+}^\alpha f \) and \( J_{b^-}^\alpha f \) denote the left-side and right-side Riemann-Liouville integrals of the order \( \alpha \) and are defined by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt \quad (0 \leq a < x),
\]

and

\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} f(t) \, dt, \quad (0 < x < b),
\]

respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by

\[
\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt
\]

and \( J_{a^+}^\alpha f(x) = J_{b^-}^\alpha f(x) = f(x) \).
In the case of $\alpha = 1$, the fractional integrals reduces to the classical integral. Recently, many authors have studied a number of inequalities by using the Riemann-Liouville fractional integrals, see [1-9,12,14-17] and the references cited therein.

Especially, in [2, 12], İmdat Işcan, Noor, and Awan proved a variant of Hermite-Hadamard and Ostrowski type inequalities which hold for the convex functions via Riemann-Liouville fractional integrals.

**Theorem 1.3.** Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on $(a, b)$ with $a < b$. If $f'' \in L([a, b])$ and $|f''|$ is convex on $[a, b]$, then we have the following inequality for fractional integrals:

$$
\left| \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(b-a)^\alpha} \left\{ J_{\frac{a+b}{2}}^{\alpha} f(a) + J_{\frac{a+b}{2}}^{\alpha} f(b) \right\} - f\left( \frac{a+b}{2} \right) \right| 
\leq \frac{(b-a)^2}{2^\alpha(\alpha+1)} \left\{ \frac{1}{2^{\alpha+3}} \beta(2, \alpha+2) + \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+4)} \right\} \left[ |f''(a)| + |f''(b)| \right].
$$

**Theorem 1.4.** Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on $(a, b)$ with $a < b$. If $f'' \in L([a, b])$ and $|f''|$ is convex on $[a, b]$, then we have the following inequality for fractional integrals:

$$
\left| \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(b-a)^\alpha} \left\{ J_{\frac{a+b}{2}}^{\alpha} f(a) + J_{\frac{a+b}{2}}^{\alpha} f(b) \right\} - f\left( \frac{a+b}{2} \right) \right| 
\leq \frac{(b-a)^2}{2^{\alpha+3/2}(\alpha+1)} \left( \frac{1}{\alpha+1} + 1 \right)^{1/2} \left\{ \left\{ 3|f''(a)|^q + |f''(b)|^q \right\}^{1/2} + \left\{ |f''(a)|^{1/q} + 3 |f''(b)|^{1/q} \right\} \right\}.
$$

In this paper, we give some generalized inequalities connected with Hermite-Hadamard-like type for differentiable functions whose derivatives in the absolute value are convex via fractional integrals.

## 2 Lemmas

Now we turn our attention to establish integral inequalities of Hermite-Hadamard and Ostrowski inequality type for convex functions via Riemann-Liouville fractional integrals, we need the lemmas below:

**Lemma 1.** Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on the interior $I^0$ of an interval $I$ such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. Then for
any \( \lambda, \mu \in \mathbb{R} \) and \( x \in [a, b] \) the following identity holds:

\[
I(f; \alpha; \lambda, \mu; x) \\
\equiv \left( \frac{\lambda}{x-a} + \frac{1-\mu}{b-x} \right) f(x) + \left( \frac{1-\lambda}{x-a} \right) f(a) + \left( \frac{\mu}{b-x} \right) f(b) \\
- \Gamma(\alpha + 1) \left\{ \frac{1}{(x-a)^{\alpha+1}} J_+^\alpha f(a) + \frac{1}{(b-x)^{\alpha+1}} J_+^\alpha f(b) \right\} \\
= \int_0^1 (t^\alpha - 1 + \lambda) f'(tx + (1-t)a) \, dt + \int_0^1 (\mu - t^\alpha) f'(tx + (1-t)b) \, dt. \tag{4}
\]

Proof. Integrating by parts and changing variable of definite integral, we have:

\[
\int_0^1 (t^\alpha - 1 + \lambda) f'(tx + (1-t)a) \, dt \\
= \lambda \frac{f(x)}{x-a} + (1-\lambda) \frac{f(a)}{x-a} - \frac{\Gamma(\alpha + 1)}{(x-a)^{\alpha+1}} J_+^\alpha f(a).
\]

Similarly, we have

\[
\int_0^1 (\mu - t^\alpha) f'(tx + (1-t)b) \, dt \\
= (1-\mu) \lambda \frac{f(x)}{b-x} + \mu \frac{f(b)}{b-x} - \frac{\Gamma(\alpha + 1)}{(b-x)^{\alpha+1}} J_+^\alpha f(b).
\]

Adding these two equalities leads to Lemma 1.

**Lemma 2.** For \( 0 \leq \xi \leq 1 \), one has

\[
(a) \int_0^1 |\xi - t^\alpha|^q \, dt \equiv \delta_1(\alpha, \xi, q) \\
\equiv \frac{\xi^q + \frac{1}{\alpha}}{\alpha} \left\{ \beta(\frac{1}{\alpha}, 1+q) + \beta(-q - \frac{1}{\alpha}, 1+q) - \beta(\xi, -q - \frac{1}{\alpha}, 1+q) \right\},
\]

\[
(b) \int_0^1 |\xi - t^\alpha|^q t \, dt \equiv \delta_2(\alpha, \xi, q) \\
\equiv \frac{\xi^q + \frac{2}{\alpha}}{\alpha} \left\{ \beta(\frac{2}{\alpha}, 1+q) + \beta(-q - \frac{2}{\alpha}, 1+q) - \beta(\xi, -q - \frac{2}{\alpha}, 1+q) \right\}. \tag{5}
\]

Proof. These equalities follows from a straightforward computation of definite integrals.
3 Some inequalities of Hermite-Hadamard and Ostrowski type

Now we turn our attention to establish new integral inequalities of Hermite-Hadamard and Ostrowski type for convex functions via fractional integrals.

**Theorem 3.1.** Let $f : I \subseteq R \to R$ be a differentiable function on the interior $I^0$ of an interval $I$ and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\lambda, \mu \in [0, 1]$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$
\left| I_{f}(f; \alpha; \lambda, \mu; x) \right| \\
\leq \delta_{p}^{\frac{1}{p}}(\alpha, 1 - \lambda, 1) \\
\times \left\{ \delta_{2}(\alpha, 1 - \lambda, 1)|f'(x)|^{q} + (\delta_{1}(\alpha, 1 - \lambda, 1) - \delta_{2}(\alpha, 1 - \lambda, 1))|f'(a)|^{q} \right\}^{\frac{1}{q}} \\
+ \delta_{p}^{\frac{1}{p}}(\alpha, \mu, 1)\left\{ \delta_{2}(\alpha, \mu, 1)|f'(x)|^{q} + (\delta_{1}(\alpha, \mu, 1) - \delta_{2}(\alpha, \mu, 1))|f'(b)|^{q} \right\}^{\frac{1}{q}}.
$$

**Proof.** Suppose that $q > 1$. From Lemma 1, the convexity of $|f'|^q$ on $[a, b]$, and the noted power-mean integral inequality, we have

$$
\left| I_{f}(f; \alpha; \lambda, \mu; x) \right| \\
\leq \int_{0}^{1} |t^{\alpha} - 1 + \lambda| |f'(tx + (1 - t)a)|^{q} dt \\
+ \int_{0}^{1} |\mu - t^{\alpha}| |f'(tx + (1 - t)b)|^{q} dt \\
\leq \left( \int_{0}^{1} |t^{\alpha} - 1 + \lambda|^{\frac{1}{p}} dt \right)^{\frac{1}{p}} \left( \int_{0}^{1} |t^{\alpha} - 1 + \lambda| |f'(tx + (1 - t)a)|^{q} dt \right)^{\frac{1}{q}} \\
+ \left( \int_{0}^{1} |\mu - t^{\alpha}|^{\frac{1}{p}} dt \right)^{\frac{1}{p}} \left( \int_{0}^{1} |\mu - t^{\alpha}| |f'(tx + (1 - t)b)|^{q} dt \right)^{\frac{1}{q}}. \quad (6)
$$

In virtue of Lemma 2, a direct calculation yields

$$
\int_{0}^{1} |t^{\alpha} - 1 + \lambda| |f'(tx + (1 - t)a)|^{q} dt \\
\leq \int_{0}^{1} |t^{\alpha} - 1 + \lambda| \left\{ t |f'(x)|^{q} + (1 - t)|f'(a)|^{q} \right\} dt \\
= \delta_{2}(\alpha, 1 - \lambda, 1)|f'(x)|^{q} + (\delta_{1}(\alpha, 1 - \lambda, 1) - \delta_{2}(\alpha, 1 - \lambda, 1))|f'(a)|^{q}, \quad (7)
$$

and

$$
\int_{0}^{1} |\mu - t^{\alpha}| |f'(tx + (1 - t)b)|^{q} dt \\
\leq \delta_{2}(\alpha, \mu, 1)|f'(x)|^{q} + (\delta_{1}(\alpha, \mu, 1) - \delta_{2}(\alpha, \mu, 1))|f'(b)|^{q}. \quad (8)
$$
By substituting the above inequalities (7) and (8) into (6), we get the desired result for $q > 1$.

Suppose that $q = 1$. From Lemma 1 and 2 it follows that

$$
\left| I_f(f; \alpha; \lambda, \mu; x) \right|
\leq \int_0^1 |t^\alpha - 1 + \lambda| |f'(tx + (1 - t)a)| dt
+ \int_0^1 |\mu - t^\alpha| |f'(tx + (1 - t)b)| dt
\leq \int_0^1 |t^\alpha - 1 + \lambda| \{t |f'(x)| + (1 - t)|f'(a)|\} dt
+ \int_0^1 |\mu - t^\alpha| \{t |f'(x)| + (1 - t)|f'(b)|\} dt
= \delta_2(\alpha, 1 - \lambda, 1) |f'(x)|
+ (\delta_1(\alpha, 1 - \lambda, 1) - \delta_2(\alpha, 1 - \lambda, 1)) |f'(a)|
+ \delta_2(\alpha, \mu, 1) |f'(x)|
+ (\delta_1(\alpha, \mu, 1) - \delta_2(\alpha, \mu, 1)) |f'(b)|,
$$

which completes the proof.

If taking $x = \frac{a + b}{2}$ in Theorem 3.1, we derive the following corollary:

**Corollary 3.1.** Let $f : I \subseteq R \to R$ be a differentiable function on the interior $I^0$ of an interval $I$ and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\lambda, \mu \in [0, 1]$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$
\left| I_f(f; \alpha; \lambda, \mu; \frac{a + b}{2}) \right|
= \frac{b - a}{2} \left| (1 - \lambda)f(a) + \mu f(b) + (1 - \mu + \lambda)f(\frac{a + b}{2}) \right|
- \left( \frac{2}{b - a} \right)^\alpha \Gamma(\alpha + 1) \left\{ J_{(\frac{a + b}{2})} f(a) + J_{(\frac{a + b}{2})} f(b) \right\}
\leq \delta_1(\alpha, 1 - \lambda, 1) \left\{ \delta_2(\alpha, 1 - \lambda, 1) |f'(\frac{a + b}{2})|^q \right. \\
+ (\delta_1(\alpha, 1 - \lambda, 1) - \delta_2(\alpha, 1 - \lambda, 1)) |f'(a)|^q \right\}^{\frac{1}{q}}
+ \delta_1^p(\alpha, \mu, 1) \left\{ \delta_2(\alpha, \mu, 1) |f'(\frac{a + b}{2})|^q \right. \\
+ (\delta_1(\alpha, \mu, 1) - \delta_2(\alpha, \mu, 1)) |f'(b)|^q \right\}^{\frac{1}{q}}.
$$

If taking $\lambda = \mu$ in Theorem 3.1, we derive the following corollary:
Corollary 3.2. Let \( f : I \subseteq R \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \) and \( \lambda \in [0, 1] \). If \( |f'|^q \) is convex on \( [a, b] \) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \lambda; x) \right| \\
\leq \delta_1^\frac{1}{p} (\alpha, 1-\lambda, 1) \\
\times \left\{ \delta_2(\alpha, 1-\lambda, 1) |f'(x)|^q + \left( \delta_1(\alpha, 1-\lambda, 1) - \delta_2(\alpha, 1-\lambda, 1) \right) |f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \delta_1^\frac{1}{p} (\alpha, 1-\lambda, 1) \left\{ \delta_2(\alpha, 1-\lambda, 1) |f'(x)|^q + \left( \delta_1(\alpha, 1-\lambda, 1) - \delta_2(\alpha, 1-\lambda, 1) \right) |f'(b)|^q \right\}^{\frac{1}{q}}.
\]

Corollary 3.3. Let \( f : I \subseteq R \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is convex on \( [a, b] \) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \frac{1}{2}; \frac{1}{2}; x) \right| \\
\leq \delta_1^\frac{1}{p} (\alpha, \frac{1}{2}, 1) \left\{ \delta_2(\alpha, \frac{1}{2}, 1) |f'(x)|^q + \left( \delta_1(\alpha, \frac{1}{2}, 1) - \delta_2(\alpha, \frac{1}{2}, 1) \right) |f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \delta_1^\frac{1}{p} (\alpha, \frac{1}{2}, 1) \left\{ \delta_2(\alpha, \frac{1}{2}, 1) |f'(x)|^q + \left( \delta_1(\alpha, \frac{1}{2}, 1) - \delta_2(\alpha, \frac{1}{2}, 1) \right) |f'(b)|^q \right\}^{\frac{1}{q}},
\]

where

\[
\delta_1(\alpha, \frac{1}{2}, 1) = \frac{1}{1 + \alpha} \left\{ \alpha \left( \frac{1}{2} \right)^{\frac{1}{q}} + \frac{1-\alpha}{2} \right\},
\]

\[
\delta_2(\alpha, \frac{1}{2}, 1) = \frac{1}{2 + \alpha} \left\{ \alpha \left( \frac{1}{2} \right)^{1+\frac{1}{q}} + \frac{2-\alpha}{4} \right\}.
\]

Corollary 3.4. Let \( f : I \subseteq R \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is convex on \( [a, b] \) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \frac{2}{3}; \frac{2}{3}; x) \right| \\
\leq \delta_1^\frac{1}{p} (\alpha, \frac{1}{3}, 1) \left\{ \delta_2(\alpha, \frac{1}{3}, 1) |f'(x)|^q + \left( \delta_1(\alpha, \frac{1}{3}, 1) - \delta_2(\alpha, \frac{1}{3}, 1) \right) |f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \delta_1^\frac{1}{p} (\alpha, \frac{2}{3}, 1) \left\{ \delta_2(\alpha, \frac{2}{3}, 1) |f'(x)|^q + \left( \delta_1(\alpha, \frac{2}{3}, 1) - \delta_2(\alpha, \frac{2}{3}, 1) \right) |f'(b)|^q \right\}^{\frac{1}{q}},
\]
where

\[
\begin{align*}
\delta_1(\alpha, \frac{1}{3}, 1) &= \frac{\alpha}{1 + \alpha} \left\{ 2\alpha \left( \frac{1}{3} \right)^{1+\frac{1}{\alpha}} + \frac{2 - \alpha}{3} \right\}, \\
\delta_1(\alpha, \frac{2}{3}, 1) &= \frac{\alpha}{1 + \alpha} \left\{ 2\alpha \left( \frac{2}{3} \right)^{1+\frac{1}{\alpha}} + \frac{1 - 2\alpha}{3} \right\}, \\
\delta_2(\alpha, \frac{1}{3}, 1) &= \frac{1}{2 + \alpha} \left\{ \alpha \left( \frac{1}{3} \right)^{1+\frac{2}{\alpha}} + \frac{4 - \alpha}{6} \right\}, \\
\delta_2(\alpha, \frac{2}{3}, 1) &= \frac{1}{2 + \alpha} \left\{ \alpha \left( \frac{2}{3} \right)^{1+\frac{2}{\alpha}} + \frac{1 - \alpha}{3} \right\}.
\end{align*}
\]

**Corollary 3.5.** Let \( f : I \subseteq R \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a,b]) \), where \( a, b \in I \) with \( a < b \). If \(|f'|^q\) is convex on \([a,b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \frac{1}{3}; \frac{1}{3}; x) \right| \\
\leq \delta_1^\frac{1}{p}(\alpha, \frac{2}{3}, 1) \left\{ \delta_2(\alpha, \frac{2}{3}, 1)|f'(x)|^q + (\delta_1(\alpha, \frac{2}{3}, 1) - \delta_2(\alpha, \frac{2}{3}, 1))|f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \delta_1^\frac{1}{p}(\alpha, \frac{1}{3}, 1) \left\{ \delta_2(\alpha, \frac{1}{3}, 1)|f'(x)|^q + (\delta_1(\alpha, \frac{1}{3}, 1) - \delta_2(\alpha, \frac{1}{3}, 1))|f'(a)|^q \right\}^{\frac{1}{q}},
\]

where \( \delta_1 \) and \( \delta_2 \) are defined as in Corollary 3.4.

**Theorem 3.2.** Let \( f : I \subseteq R \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a,b]) \), where \( a, b \in I \) with \( a < b \) and \( \lambda, \mu \in [0,1] \). If \(|f'|^q\) is convex on \([a,b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \lambda, \mu; x) \right| \\
\leq \left\{ \delta_2(\alpha, 1 - \lambda, q)|f'(x)|^q + (\delta_1(\alpha, 1 - \lambda, q) - \delta_2(\alpha, 1 - \lambda, q))|f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \left\{ \delta_2(\alpha, q)|f'(x)|^q + (\delta_1(\alpha, q) - \delta_2(\alpha, q))|f'(b)|^q \right\}^{\frac{1}{q}}.
\]

**Proof.** Suppose that \( q > 1 \). By Lemma 1, the convexity of \(|f'|^q\) on \([a,b]\),
and Hölder’s integral inequality, it follows that
\[
|I_f(f; \alpha; \lambda, \mu; x)| \leq \int_0^1 |t^\alpha - 1 + \lambda| |f'(tx + (1 - t)a)| dt \\
+ \int_0^1 |\mu - t^\alpha| |f'(tx + (1 - t)b)| dt \\
\leq \left( \int_0^1 dt \left( \int_0^1 |t^\alpha - 1 + \lambda|^q |f'(tx + (1 - t)a)|^q dt \right) \right)^{\frac{1}{q}} \\
+ \left( \int_0^1 dt \left( \int_0^1 |\mu - t^\alpha|^q |f'(tx + (1 - t)b)|^q dt \right) \right)^{\frac{1}{q}} \\
= \left( \int_0^1 |t^\alpha - 1 + \lambda|^q |f'(tx + (1 - t)a)|^q dt \right)^{\frac{1}{q}} \\
+ \left( \int_0^1 |\mu - t^\alpha|^q |f'(tx + (1 - t)b)|^q dt \right)^{\frac{1}{q}}.
\] (9)

In virtue of Lemma 2, a direct calculation yields
\[
\int_0^1 |t^\alpha - 1 + \lambda|^q |f'(tx + (1 - t)a)|^q dt \\
\leq \int_0^1 |t^\alpha - 1 + \lambda|^q \{t |f'(x)| + (1 - t) |f'(a)| \}^q dt \\
= \delta_2(\alpha, 1 - \lambda, q) |f'(x)|^q + (\delta_1(\alpha, 1 - \lambda, q) - \delta_2(\alpha, 1 - \lambda, q)) |f'(a)|^q, \tag{10}
\]
and
\[
\int_0^1 |\mu - t^\alpha|^q |f'(tx + (1 - t)b)|^q dt \\
\leq \delta_2(\alpha, \mu, q) |f'(x)|^q + (\delta_1(\alpha, \mu, q) - \delta_2(\alpha, \mu, q)) |f'(b)|^q. \tag{11}
\]

By substituting the above inequalities (10) and (11) into (9), we get the desired result for $q > 1$.

For $q = 1$, from Lemma 1 and 2 it follows that
\[
\left| I_f(f; \alpha; \lambda, \mu) \right| \leq \int_0^1 |t^\alpha - 1 + \lambda| \{t |f'(x)| + (1 - t) |f'(a)| \} dt \\
+ \int_0^1 |\mu - t^\alpha| \{t |f'(x)| + (1 - t) |f'(b)| \} dt \\
= \delta_2(\alpha, 1 - \lambda, 1) |f'(x)| + (\delta_1(\alpha, 1 - \lambda, 1) - \delta_2(\alpha, 1 - \lambda, 1)) |f'(a)| \\
+ \delta_2(\alpha, \mu, 1) |f'(x)| + (\delta_1(\alpha, \mu, 1) - \delta_2(\alpha, \mu, 1)) |f'(b)|,
\]
which completes the proof.
Corollary 3.6. Let \( f : I \subseteq R \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \) and \( \lambda, \mu \in [0, 1] \). If \( |f'|^q \) is convex on \([a, b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \lambda, \mu; \frac{a + b}{2}) \right| \\
= \frac{b - a}{2} \left| (1 - \lambda)f(a) + \mu f(b) + (1 - \mu + \lambda)f\left(\frac{a + b}{2}\right) \right| \\
- \left( \frac{2}{b - a} \right)^\alpha \Gamma(\alpha + 1) \left\{ f_{\left(\frac{2\alpha + 1}{\alpha + 1}\right)}' a + f_{\left(\frac{2\alpha + 1}{\alpha + 1}\right)}' (b) \right\} \\
\leq \left\{ \delta_2(\alpha, 1 - \lambda, q)|f'(\frac{a + b}{2})|^q \\
+ (\delta_1(\alpha, 1 - \lambda, q) - \delta_2(\alpha, 1 - \lambda, q))|f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \left\{ \delta_2(\alpha, \mu, q)|f'(\frac{a + b}{2})|^q \\
+ (\delta_1(\alpha, \mu, q) - \delta_2(\alpha, \mu, q))|f'(b)|^q \right\}^{\frac{1}{q}}.
\]

If taking \( \lambda = \mu \) in Theorem 3.1, we derive the following corollary:

Corollary 3.7. Let \( f : I \subseteq R \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \) and \( \lambda \in [0, 1] \). If \( |f'|^q \) is convex on \([a, b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \lambda, \lambda; x) \right| \\
\leq \left\{ \delta_2(\alpha, 1 - \lambda, q)|f'(x)|^q + (\delta_1(\alpha, 1 - \lambda, q) - \delta_2(\alpha, 1 - \lambda, q))|f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \left\{ \delta_2(\alpha, \lambda, q)|f'(x)|^q + (\delta_1(\alpha, \lambda, q) - \delta_2(\alpha, \lambda, q))|f'(b)|^q \right\}^{\frac{1}{q}}.
\]

Corollary 3.8. Let \( f : I \subseteq R \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is convex on \([a, b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \frac{1}{2}, \frac{1}{2}; x) \right| \\
\leq \left\{ \delta_2(\alpha, \frac{1}{2}, q)|f'(x)|^q + (\delta_1(\alpha, \frac{1}{2}, q) - \delta_2(\alpha, \frac{1}{2}, q))|f'(a)|^q \right\}^{\frac{1}{q}} \\
+ \left\{ \delta_2(\alpha, \frac{1}{2}, q)|f'(x)|^q + (\delta_1(\alpha, \frac{1}{2}, q) - \delta_2(\alpha, \frac{1}{2}, q))|f'(b)|^q \right\}^{\frac{1}{q}}.
\]
Corollary 3.9. Let \( f : I \subseteq R \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is convex on \([a, b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \frac{2}{3}; \frac{2}{3}; x) \right| \leq \left\{ \delta_2(\alpha; \frac{1}{3}, q)|f'(x)|^q + (\delta_1(\alpha; \frac{1}{3}, q) - \delta_2(\alpha; \frac{1}{3}, q))|f'(a)|^q \right\}^{\frac{1}{q}} + \left\{ \delta_2(\alpha; \frac{2}{3}, q)|f'(x)|^q + (\delta_1(\alpha; \frac{2}{3}, q) - \delta_2(\alpha; \frac{2}{3}, q))|f'(b)|^q \right\}^{\frac{1}{q}}.
\]

Theorem 3.3. Let \( f : I \subseteq R \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \) and \( \lambda, \mu \in [0, 1] \). If \( |f'|^q \) is convex on \([a, b]\) for \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I_f(f; \alpha; \lambda; \mu; x) \right| \leq \delta_1^\frac{1}{q}(\alpha, 1 - \lambda, p)\left\{ \frac{|f'(x)|^q + |f'(a)|^q}{2} \right\}^{\frac{1}{q}} + \delta_1^\frac{1}{q}(\alpha, \mu, p)\left\{ \frac{|f'(x)|^q + |f'(b)|^q}{2} \right\}^{\frac{1}{q}}.
\]

Proof. Suppose that \( q > 1 \). From Lemma 1, the convexity of \(|f'|^q\) on \([a, b]\), and the Hölder’s integral inequality, we have

\[
\left| I_f(f; \alpha; \lambda; \mu; x) \right| \leq \left( \int_0^1 |t^\alpha - 1 + \lambda|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1 - t)a)|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 |\mu - t^\alpha|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \leq \delta_1^\frac{1}{q}(\alpha, 1 - \lambda, p)\left( \int_0^1 \{ t|f'(x)|^q + (1 - t)|f'(a)|^q \} dt \right)^{\frac{1}{q}} + \delta_1^\frac{1}{q}(\alpha, \mu, p)\left( \int_0^1 \{ t|f'(x)|^q + (1 - t)|f'(b)|^q \} dt \right)^{\frac{1}{q}} \leq \delta_1^\frac{1}{q}(\alpha, 1 - \lambda, p)\left( \frac{|f'(x)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \delta_1^\frac{1}{q}(\alpha, \mu, p)\left( \frac{|f'(x)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}},
\]

which completes the proof.
References


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